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DOCTORAL THESIS

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Problems in discrete geometry

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I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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In Prague, 18th June 2015 Zuzana Patáková

Název práce: Problémy diskrétní geometrie

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Abstrakt: V této práci se věnujeme třem různým problémům z oblasti diskrétní geometrie. Společným pojítkem těchto problémů je, že jejich řešení využívá algebraické metody.

První problém se zabývá tzv. polynomiální metodou, která konečnou množinu bodů rozdělí pomocí nulové množinu polynomu. Limitujícím faktorem této metody je, co dělat s body, které leží v nulové množině získaného polynomu? V práci představujeme obecnou verzi, která řeší popsanou situaci, a jako aplikaci uvádíme nový algoritmus pro tzv. *semialgebraický range searching problém.*

V druhé části práce se věnujeme studiu Ramseyových funkcí semialgebraických predikátů. Conlon, Fox, Pach, Sudakov a Suk zkonstruovali první příklady semialgebraických predikátů s Ramseyovou funkcí zespoda odhadnutou věžovitou funkcí. My snížíme dimenzi příslušného prostoru v jejich konstrukci a jako důsledek ukážeme novou geometrickou větu Ramseyova typu s velkou Ramseyovou funkcí.

V poslední části se zabýváme samodlážditelnými simplexy. Simplex S je *k-samodlážditelný*, pokud je sjednocením k navzájem shodných simplexů s disjunktními vnitřky, které jsou navíc podobné simplexu S. V práci ukážeme, že v dimenzi čtyři mohou k-samodlážditelné simplexy existovat jen pro k tvaru m^2, kde $m \geq 1$ je celé číslo.

Klíčová slova: polynomiální dělení, Ramseyova funkce, samodlážditelný simplex

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Abstract: This thesis studies three different questions from discrete geometry. A common theme for these problems is that their solution is based on algebraic methods.

First part is devoted to the polynomial partitioning method, which partitions a given finite point set using the zero set of a suitable polynomial. However, there is a natural limitation of this method, namely, what should be done with the points lying in the zero set? Here we present a general version dealing with the situation and as an application, we provide a new algorithm for the *semialgebraic range searching problem.*

In the second part we study Ramsey functions of semialgebraic predicates. Conlon, Fox, Pach, Sudakov, and Suk constructed the first examples of semialgebraic predicates with the Ramsey function bounded from below by a tower function. We reduce the dimension of the ambient space in their construction and as a consequence, we provide a new geometric Ramsey-type theorem with a large Ramsey function.

Last part is devoted to reptile simplices. A simplex S is *k-reptile* if it can be tiled by k simplices with disjoint interiors that are all mutually congruent and similar to S. We show that four-dimensional k-reptile simplices can exist only for $k = m^2$, where $m \geq 1$ is an integer.

Keywords: polynomial partitions, Ramsey function, reptile simplex

Contents

Introduction

Discrete geometry is a mathematical discipline studying combinatorial properties of discrete geometric objects. There are many problems concerning objects such as points, lines, or triangles usually living in a plane or more generally, in d-dimensional Euclidean space. While such problems are often easy to state, some of them turn out to be very difficult and the answer is not known for decades. When solving such problems, it might be useful to use some tools from other fields of mathematics, such as number theory or algebra, or to come up with some deep underlying theory.

In this thesis we focus on three different problems in discrete geometry. A common theme for these problems is that their solution is based on linear algebra or algebraic methods.

The thesis is based on the following four papers:

1. H. Kaplan, J. Matoušek, Z. Safernová, and M. Sharir. Unit distances in three dimensions. *Comb. Probab. Comput.*, 21(4):597–610, 2012.

2. J. Matoušek and Z. Patáková. Multilevel Polynomial Partitions and Simplified Range Searching. *Discrete Comput. Geom.*, 54(1):22–41, 2015.

3. M. Eliáš, J. Matoušek, E. Roldán Pensado, and Z. Safernová. Lower bounds on geometric Ramsey functions. *SIAM J. Discrete Math.*, 28(4):1960–1970, 2014.

4. J. Kynčl and Z. Patáková. On the nonexistence of k-reptile simplices in $\mathbb{R}^3$ and $\mathbb{R}^4$. In preparation. Extended abstract appeared in *The Seventh European Conference on Combinatorics, Graph Theory and Applications*, 191–196, CRM Series, 16, Ed. Norm., Pisa, 2013.

Chapter 1 is based on a small part of (1) and slightly modified version of (2). However, Sections 1.3.2 and 1.10 are new, Theorem 1.4 and Lemmas 1.14, 1.18 as well. Chapter 2 contains the paper (3) while paper (4) will be based on Chapter 3.

Overview of the results

Chapter 1 is devoted to the polynomial partitioning method of Guth and Katz [GK15], which has numerous applications in discrete and computational geometry. It partitions a given n-point set $P \subset \mathbb{R}^d$ using the zero set $Z(f)$ of a suitable d-variate polynomial f. Applications of this result are often complicated by the problem, what should be done with the points of P lying within $Z(f)$? A natural approach is to partition these points with another polynomial and continue further in a similar manner.

We will follow this plan and first we introduce a second partitioning polynomial. Then, as a main result, we provide a polynomial partitioning method with up to d

polynomials in dimension d, which allows for a complete decomposition of the given point set. We apply it to obtain a new algorithm for the *semialgebraic range searching problem*. Our algorithm has running time bounds similar to a recent algorithm by Agarwal, Matoušek, and Sharir [AMS13], but it is simpler both conceptually and technically.

In Chapter 2 we continue in a sequence of recent works studying Ramsey functions for semialgebraic predicates in $\mathbb{R}^d$. A *k-ary semialgebraic predicate* $\Phi(x_1, \ldots, x_k)$ on $\mathbb{R}^d$ is a Boolean combination of polynomial equations and inequalities in the kd coordinates of k points $x_1, \ldots, x_k \in \mathbb{R}^d$. A sequence $P = (p_1, \ldots, p_n)$ of points in $\mathbb{R}^d$ is called Φ-*homogeneous* if either $\Phi(p_{i_1}, \ldots, p_{i_k})$ holds for all choices $1 \leq i_1 < \cdots < i_k \leq n$, or it does not hold for any such choice. The Ramsey function $R_\Phi(n)$ is the smallest N such that every point sequence of length N contains a Φ-homogeneous subsequence of length n.

Conlon, Fox, Pach, Sudakov, and Suk [CFP$^+$14] constructed the first examples of semialgebraic predicates with the Ramsey function bounded from below by a tower function of arbitrary height: for every $k \geq 4$, they exhibit a k-ary Φ in dimension 2^{k-4} with R_Φ bounded below by a tower of height $k-1$. We reduce the dimension in their construction, obtaining a k-ary semialgebraic predicate Φ on $\mathbb{R}^{k-3}$ with R_Φ bounded below by a tower of height $k-1$.

We also provide a natural geometric Ramsey-type theorem with a large Ramsey function. We call a point sequence P in $\mathbb{R}^d$ *order-type homogeneous* if all $(d+1)$-tuples in P have the same orientation. Every sufficiently long point sequence in general position in $\mathbb{R}^d$ contains an order-type homogeneous subsequence of length n, and the corresponding Ramsey function has recently been studied in several papers. Together with a recent work of Bárány, Matoušek, and Pór, our results imply a tower function of $\Omega(n)$ of height d as a lower bound, matching an upper bound by Suk up to the constant in front of n.

In Chapter 3 we study reptile simplices. A d-dimensional simplex S is called a *k-reptile* (or a *k-reptile simplex*) if it can be tiled by k simplices with disjoint interiors that are all mutually congruent and similar to S. For $d = 2$, triangular k-reptiles exist if and only if k has the form $a^2, 3a^2$ or $a^2 + b^2$ and they have been completely characterized by Snover, Waiveris, and Williams. On the other hand, the only k-reptile simplices that are known for $d \geq 3$, have $k = m^d$, where m is a positive integer. We substantially simplify the proof by Matoušek and the author that for $d = 3$, k-reptile tetrahedra can exist only for $k = m^3$. We also prove a weaker analogue of this result for $d = 4$ by showing that four-dimensional k-reptile simplices can exist only for $k = m^2$.

Chapter 1

Multilevel polynomial partitions and simplified range searching

1.1 Introduction

Polynomial partitions. Since the late 1980s, numerous problems in discrete and computational geometry have been solved by geometric divide-and-conquer method, where a suitable partition of space is used to subdivide a geometric problem into simpler subproblems.

The earliest, and most widely applied, kinds of such partitions are *cuttings*, based mainly on ideas of Clarkson (e.g., [Cla87]) and Haussler and Welzl [HW87]. See, e.g., [Cha05] for a survey of cuttings and their applications.

Using cuttings as the main tool, another kind of space partition, called *simplicial partitions*, was introduced in [Mat92] (and further improved by Chan [Cha12]). Given an n-point set $P \subset \mathbb{R}^d$ and a parameter $r > 1$, a simplicial $\frac{1}{r}$-partition is a collection of simplices (of dimensions 0 through d), such that each of them contains at most n/r points of P and together they cover P. In Chan's version, they can also be assumed to be pairwise disjoint.

Let us introduce the following convenient terminology: a set A *crosses* a set B if A intersects B but does not contain it. The main parameter of a simplicial partition is the maximum number of simplices of the partition that can be simultaneously crossed by a hyperplane (or, equivalently, by a halfspace). One can construct simplicial partitions where this number is bounded by $O(r^{1-1/d})$ [Mat92, Cha12], which is asymptotically optimal in the worst case (throughout this chapter, we consider the space dimension d as a constant, and the implicit constants in asymptotic notation may depend on it, unless explicitly stated otherwise).

Simplicial partitions work mostly fine for problems involving points and hyperplanes in $\mathbb{R}^d$. However, they are much less useful if hyperplanes are replaced by lower-dimensional objects—such as lines—or curved objects—such as spheres—or other hypersurfaces.

Guth and Katz [GK15] invented a new kind of partitions, called *polynomial partitions*, which overcome these drawbacks to some extent. The most striking application of polynomial partitions so far is probably still the original one—a solution of Erdős' problem of distinct distances [GK15] (also see Guth [Gut15] for a simplified but weaker version of the main result of [GK15]), but a fair number of other applications have been found since then: see the works of Solymosi and Tao [ST12], Zahl [Zah13], Kaplan, Matoušek and Sharir [KMS12], Kaplan, Matoušek, Safernová, and Sharir [KMSS12], Zahl [Zah12], Wang, Yang, and Zhang [WYZ13], Agarwal, Matoušek, and Sharir [AMS13], Sharir, Sheffer, and Zahl [SSZ15], and Sharir and Solomon [SS14] (our list is most likely incomplete and we apologize for omissions).

Given an n-point set $P \subset \mathbb{R}^d$ and a parameter $r > 1$, we say that a nonzero polynomial $f \in \mathbb{R}[x_1, \ldots, x_d]$ is a $\frac{1}{r}$*-partitioning polynomial* for P if none of the connected components of $\mathbb{R}^d \setminus Z(f)$ contains more than n/r points of P.

Guth and Katz [GK15] proved the following theorem:

Theorem 1.1 ([GK15])**.** *For every finite point set $P \subseteq \mathbb{R}^d$ and every $r > 1$ there exists a $\frac{1}{r}$-partitioning polynomial of degree $O(r^{1/d})$.*

From the results of real algebraic geometry on the complexity of arrangements of zero sets of polynomials (see [BPR03]) it follows that any hyperplane h intersects at most $O(r^{1-1/d})$ components of $\mathbb{R}^d \setminus Z(f)$, and hence any halfspace crosses at most $O(r^{1-1/d})$ components of $\mathbb{R}^d \setminus Z(f)$. Moreover, using a more recent result of Barone and Basu [BB12] discussed below, one obtains that an algebraic variety X of dimension k defined by polynomials of constant-bounded degrees crosses at most $O(r^{k/d})$ components of $\mathbb{R}^d \setminus Z(f)$. In this respect, polynomial partitions match the performance of simplicial partitions concerning hyperplanes and give a crucial advantage for other varieties. However, they still leave an important issue open: namely, what should be done with the *exceptional set* $P^* := P \cap Z(f)$ that ends up lying within the zero set of the partitioning polynomial.

Multilevel polynomial partitions. At first sight, it may seem that this issue can be remedied, say, by a suitable perturbation of the polynomial f. However, if all of P lies on a line in $\mathbb{R}^d$, say, then a degree-D polynomial can partition it into at most $D + 1$ pieces, and so if we want all of P to be partitioned into pieces of size n/r, we will need degree about r, as opposed to $r^{1/d}$ in the Guth–Katz polynomial partition theorem.

A natural idea is to partition the exceptional set P^* further by another polynomial g such that $Z(f, g) := Z(f) \cap Z(g)$ has dimension at most $d-2$. If $Z(f, g)$ again contains many points of P^*, we would like to partition them further by a third polynomial h with $\dim Z(f, g, h) \leq d - 3$, and so on.

This approach encounters several technical difficulties, and so far it has been realized only up to the second partitioning polynomial g in [Zah13] and [KMSS12] (also see [Zah12]). We note that our formulation is different from the one in [KMSS12].

Theorem 1.2. *Let $f \in \mathbb{R}[x_1, \ldots, x_d]$ be an irreducible polynomial of degree $D \geq 1$. For every finite point set $P \subseteq Z(f)$ and every $r > 1$ there exists a $\frac{1}{r}$-partitioning polynomial for P of degree $O\left(D + \left(\frac{r}{D}\right)^{1/(d-1)}\right)$ which is co-prime with f.*

Usually in the applications, we think about n (number of points) as very big and the typical choice of r depends on n. Note that for $r = \Omega(D^d)$, the term $O\left(\left(\frac{r}{D}\right)^{1/(d-1)}\right)$ dominates.

Since f is irreducible and g coprime with f, it follows that $\dim Z(f, g) \leq d-2$ (we refer to Section 1.3.2 or for $d = 3$ to [KMSS12]).

Our main result is the following multilevel partition theorem.

Theorem 1.3. *For every integer $d > 1$ there is a constant K such that the following hold. Given an n-point set $P \subset \mathbb{R}^d$ and a parameter $r > 1$, there are numbers $r_1, r_2, \dots, r_d \in [r, r^K]$, positive integers $t_1, t_2, \dots, t_d$, a partition*

$$P = P^* \cup \bigcup_{i=1}^{d} \bigcup_{j=1}^{t_i} P_{ij}$$

of P into disjoint subsets, and for every i, j, a connected set $S_{ij} \subseteq \mathbb{R}^d$ containing P_{ij}, such that $|P_{ij}| \le n/r_i$ for all i, j, $|P^*| \le r^K$, and the following hold:

(i) *If $h \in \mathbb{R}[x_1, \dots, x_d]$ is a polynomial of degree bounded by a constant D_0, and $X = Z(h)$ is its zero set, then, for every $i = 1, 2, \dots, d$, the number of the S_{ij} crossed by X is at most $O\left(r_i^{1-1/d}\right)$, with the implicit constant also depending on D_0.*

(ii) *If X is an algebraic variety in $\mathbb{R}^d$ of dimension at most $k \le d-2$ defined by polynomials of degree bounded by a constant D_0, then, for every $i = 1, 2, \dots, d$, the number of the S_{ij} crossed by X is bounded by $O\left(r_i^{1-1/(k+1)}\right)$.*

We will need only part (i), while part (ii) is stated for possible future use, since it can be handled with very little extra work.

We note that the sets S_{ij} have special form, namely, they correspond to cells of an arrangement of zero sets of certain polynomials.

For some values of the parameter r we get, by a simple modification of the proof, the following variant of Theorem 1.3.

Theorem 1.4. *Let $r > 1$ be a parameter satisfying $r^K \le n$, where K and n are as in Theorem 1.3. Then there is a partition of n-point set P satisfying all the conditions as the partition from Theorem 1.3 such that, moreover, P^* is empty.*

Related work. The problem concerning the exceptional set P^* in a single-level polynomial partition has been addressed in various ways in the literature.

In one of the theorems in Agarwal et al. [AMS13], P^* is forced to be at most of a constant size, by an infinitesimal perturbation of P. However, this strategy cannot be used in incidence problems, for example, where a perturbation destroys the structure of interest. Moreover, for algorithmic purposes, known methods of infinitesimal perturbation are applicable with a reasonable overhead only for constant values of r.

Solymosi and Tao [ST12] handle the exceptional set essentially by projecting it to a hyperplane. This yields a $(d-1)$-dimensional problem, which is handled recursively. Their method allows them to deal only with constant values of r, and consequently it yields bounds that are suboptimal by factors of n^ε (where $\varepsilon > 0$ is arbitrarily small but fixed number).

Another variant of the strategy of projecting P^* to a hyperplane was used in [AMS13]; there r could be chosen as a small but fixed power of n, leading to only polylogarithmic extra factors, as opposed to n^ε with constant r. However, the resulting algorithm and proof are complicated, since one has to keep track of several parameters and solve a tricky recursion.

Our proof of Theorem 1.3 also involves a projection trick, but the projection is encapsulated in the proof and simple to analyze, and when applying the theorem we can work in the original space all the time.

In this chapter we apply an algorithmic enhancement of Theorem 1.3 to recover the main result of Agarwal et al. [AMS13] in a way that is simpler both conceptually and technically.

It is worth mentioning that simultaneously with us, two groups of researchers obtained results concerning multilevel polynomial partitions, which partially overlap with ours. Fox, Pach, Sheffer, Suk, and Zahl [FPS⁺14] as well as Basu and Sombra [BS14], obtained results similar to our key lemma (Lemma 1.13), but with different proofs. However, the result of Basu and Sombra works just for varieties of codimension two and hence it cannot be used for our range searching algorithm. On the other hand, Fox et al. have no restriction on the dimension of the variety, but they have to assume that the variety is irreducible. The important feature of our method is that we are able to avoid computing irreducible components which is crucial from algorithmic point of view. For more details we refer to the discussion in Section 1.9.

Range searching with semialgebraic sets. Here we consider a basic and long-studied question in computational geometry.

Let P be a set of n points in $\mathbb{R}^d$ and let Γ be a family of geometric "regions," called *ranges*, in $\mathbb{R}^d$. For example, Γ can be the set of all axis-parallel boxes, balls, simplices, or cylinders, or the set of all intersections of pairs of ellipsoids. In the Γ-*range searching* problem, we want to preprocess P into a data structure so that the number of points of P lying in a query range $\gamma \in \Gamma$ can be counted efficiently. More generally, we may be given a weight function on the points in P and we ask for the cumulative weight of the points in $P \cap \gamma$ (our result applies in this more general setting as well). We consider the *low-storage* variant of Γ-range searching, where the data structure is allowed to use only linear or near-linear storage, and the goal is to make the query time as small as possible.

We study *semialgebraic range searching*, where Γ is a set of constant-complexity semialgebraic sets. We recall that a *semialgebraic set* is a subset of $\mathbb{R}^d$ obtained from a finite number of sets of the form $\{x \in \mathbb{R}^d \mid g(x) \geq 0\}$, where g is a d-variate polynomial with integer coefficients, by Boolean operations (unions, intersections, and complementations). Specifically, let $\Gamma_{d,D,s}$ denote the family of all semialgebraic sets in $\mathbb{R}^d$ defined by at most s polynomial inequalities of degree at most D each. By *semialgebraic range searching* we mean $\Gamma_{d,D,s}$-range searching for some parameters d, D, s.

This problem and various special cases of it have been studied in many papers. We refer to [AE98, Mat95] for background on range searching and to [AMS13] for a more detailed discussion of the problem setting and previous work.

The main result of [AMS13] is as follows.

Theorem 1.5. *Let d, D_0, s, and $\varepsilon > 0$ be constants. Then the $\Gamma_{d,D_0,s}$-range searching problem for an arbitrary n-point set in $\mathbb{R}^d$ can be solved with $O(n)$ storage, $O\left(n^{1+\varepsilon}\right)$ expected preprocessing time, and $O\left(n^{1-1/d}\log^B n\right)$ query time, where B is a constant depending on d, D_0, s and ε.*

As announced, here we provide a new and simpler proof. Basically we apply Theorem 1.3, but for the algorithmic application, we need to amend it with an algorithmic part, essentially asserting that the construction in Theorem 1.3 can be executed in time depending polynomially on r and linearly on n (we again stress that d is taken as a constant). Moreover, we need that the S_{ij} can be handled algorithmically—they are semialgebraic sets of controlled complexity. We will use the real RAM model of computation where we can compute exactly with arbitrary real numbers and each arithmetic operation is executed in unit time.

A precise statement is as follows.

Theorem 1.6 (Algorithmic enhancement of Theorem 1.3)**.** *Given $P \subset \mathbb{R}^d$ and r as in Theorem 1.3, one can compute the sets P^*, P_{ij}, and S_{ij} in time $O\left(nr^C\right)$, where $C = C(d)$ is a constant. Moreover, for every i, the number t_i of the P_{ij} is $t_i = O\left(r^C\right)$, and each S_{ij} is a semialgebraic set defined by at most $O\left(r^C\right)$ polynomial inequalities of maximum degree $O\left(r^C\right)$. For every $i = 1, 2, \ldots, d$, every range $\gamma \in \Gamma_{d,D_0,s}$ crosses at most $O\left(r_i^{1-1/d}\right)$ of the S_{ij}, with the constant of proportionality depending on d, D_0, s.*

1.2 Algebraic preliminaries I

An *affine real algebraic variety* V is a subset of some $\mathbb{R}^d$ that can be expressed as $V = Z(f_1, \ldots, f_m)$, that is, the set of common zeros of finitely many polynomials $f_1, \ldots, f_m \in \mathbb{R}[x_1, \ldots, x_d]$. For an *affine complex algebraic variety*, $\mathbb{R}$ is replaced with $\mathbb{C}$ (the complex numbers).

Let $\mathrm{P}\mathbb{C}^d$ denote some d-dimensional projective space over $\mathbb{C}$. The *projectivization* of a point $p = (p_1, \ldots, p_d) \in \mathbb{C}^d$ is obtained by passing to *homogeneous coordinates*, and by assigning $p^\dagger = (1, p_1, \ldots, p_d)$ to p, where all nonzero scalar multiples of $p^\dagger$ are identified with $p^\dagger$. We use the standard notation $[1\colon p_1\colon \ldots : p_d]$ for projective coordinates in order to distinguish them from the affine ones.

Similarly as in the affine case, a *projective complex variety* V is a subset of some $\mathrm{P}\mathbb{C}^d$ that can be expressed as the set of common zeros of finitely many *homogeneous* polynomials $f_1, \ldots, f_m \in \mathbb{C}[x_0, x_1, \ldots, x_d]$, where a nonzero polynomial is called homogeneous, if all its terms have the same degree.

If $f_1, \ldots, f_s \in \mathbb{C}[x_0, \ldots, x_d]$ are homogeneous polynomials determining a projective variety V, then $V \cap \mathbb{C}^d$ is an affine variety given by the set of polynomials $f_i(1, x_1, \ldots, x_d)$, $1 \le i \le s$. By a *linear subspace* of $\mathrm{P}\mathbb{C}^d$ we mean a linear variety, that is, a zero set of a finite collection of linear homogeneous polynomials.

As in the introduction, we will use $Z(f)$ for the real zeros of a (real) polynomial $f \in \mathbb{R}[x_1, \ldots, x_d]$, while $Z_{\mathbb{C}}(f)$ is the set of all zeros of a complex or real polynomial in $\mathbb{C}^d$. For a real polynomial f we have $Z(f) = Z_{\mathbb{C}}(f) \cap \mathbb{R}^d$.

A nonempty (affine or projective) complex variety V is called *irreducible* if it cannot be written as the union of two proper complex subvarieties, and similarly for real varieties. The empty set is not considered to be irreducible. Note that $Z(f)$ can be irreducible over $\mathbb{R}$ even if $Z_{\mathbb{C}}(f)$ is reducible over $\mathbb{C}$. An easy example is the variety $V(x^2+y^2)$. It is well known that every nonempty variety can be uniquely decomposed into a finite number of irreducible components, none containing another.

For a complex (affine or projective) variety V, we will use the notions of *dimension* $\dim V$ and *degree* $\deg V$. These can be defined in several equivalent ways. We refer to the literature such as [CLO07, Har92, Har77] for rigorous treatment. Here we just recall a rather intuitive definition and state the properties we will actually use.

The dimension of $V \subseteq \mathbb{C}^d$ can be defined as the largest k such that a generic $(d-k)$-dimensional complex affine subspace F of $\mathbb{C}^d$ intersects V in finitely many points, and the degree is the number of intersections (which is the same for all generic F). To explain the meaning of "generic", let us consider only the subspaces $F = F(a)$ that can be expressed by the equations $x_{i+d-k} = a_{i0} + \sum_{j=1}^{d-k} a_{ij}x_j$, $i = 1, \ldots, k$, for some $a = (a_{ij})_{i=1}^{k}{}_{j=0}^{d-k} \in \mathbb{C}^{k(d-k+1)}$. The $F(a)$ being generic means that the point a does not lie in the zero set of a certain nonzero polynomial (depending on V). In particular, almost all subspaces F in the sense of measure are generic. The dimension and degree of $V \subseteq \mathrm{P}\mathbb{C}^d$ can be defined completely analogically, we just consider a generic $(d-k)$-dimensional linear subspace of $\mathrm{P}\mathbb{C}^d$. We note that the dimension of $\mathbb{C}^d$ and $\mathrm{P}\mathbb{C}^d$ is d and its degree is 1, respectively.

If $V \subset \mathbb{C}^d$ (or $V \subset \mathrm{P}\mathbb{C}^d$) is the zero set of a single squarefree polynomial f, then $\deg V = \deg f$. When considering a zero set of a single polynomial we will, without loss of generality, always assume that the polynomial we deal with is squarefree.

For a real algebraic variety V, the definition with a generic affine subspace does not quite make sense, and in real algebraic geometry, the dimension is usually defined, for the more general class of semialgebraic sets, as the largest k such that V contains the image of a k-dimensional open cube under an injective semialgebraic map; see [BCR98, BPR03]. An equivalent way of defining the dimension of a real algebraic variety V uses Krull dimension[1] of the *coordinate ring* $\mathbb{R}[x_1, \ldots, x_d]/\mathbf{I}(V)$, where $\mathbf{I}(V)$ is the ideal of all real polynomials vanishing on V; see [BCR98, Cor. 2.8.9] for this equivalence. For complex case the dimension defined via generic affine subspaces coincides with the Krull dimension of the coordinate ring $\mathbb{C}[x_1, \ldots, x_d]/\mathbf{I}_{\mathbb{C}}(V)$; see [Har92, Chapter 11].

Let $V \subseteq \mathbb{C}^d$ be an affine variety. The smallest projective variety containing V is called *projective closure of* V and denoted by $\overline{V}$. By [CLO07, Exercise 8.4.9], $V \subseteq \mathbb{C}^d$ is irreducible if and only if $\overline{V} \subseteq P\mathbb{C}^d$ is irreducible. Moreover, by [CLO07, Thm. 9.3.12] $\dim V = \dim \overline{V}$ and $\deg V = \deg \overline{V}$.

We will also need the following fact, which is apparently standard (for example, it is mentioned without proof as Remark 13 in [BB13]), although so far we have not been able to locate an explicit reference (Whitney [Whi57, Lemma 8] proves a similar statement, but he uses definitions that are not standard in the current literature).

Lemma 1.7. *Let $V \subseteq \mathbb{C}^d$ be a complex variety. Then $V \cap \mathbb{R}^d$ is a real variety and $\dim(V \cap \mathbb{R}^d) \le \dim V$.*

This is perhaps not as obvious as it may seem, because if we identify $\mathbb{C}^d$ with $\mathbb{R}^{2d}$ in the usual way, then topologically, a k-dimensional complex variety V has (real) dimension $2k$.

Sketch of proof. If $V = Z_{\mathbb{C}}(f_1, \ldots, f_m)$ for $f_1, \ldots, f_m \in \mathbb{C}[x_1, \ldots, x_d]$, then

$$V \cap \mathbb{R}^d = Z(f_1\overline{f}_1, \ldots, f_m\overline{f}_m),$$

where the bar denotes complex conjugation. Each $f_i\overline{f}_i$ is a real polynomial, and so $V \cap \mathbb{R}^d$ is a real variety.

The inequality for the dimensions can be checked, for example, by employing the definition of the dimensions via the Hilbert function (see, e.g., [CLO07]), which is well known to be equivalent to the Krull dimension definition. Indeed, if $f \in \mathbb{C}[x_1, \ldots, x_d]$

[1]Krull dimension of a ring R is the largest n such that there exists a chain $I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_n$ of nested prime ideals in R.

is a complex polynomial of degree at most D vanishing on V, we can write $f = f_1 + if_2$, where $f_1, f_2 \in \mathbb{R}[x_1, \ldots, x_d]$ correspond to the real and complex parts of coefficients of f, respectively. Then $\deg f_1$ and $\deg f_2$ are at most D and both f_1 and f_2 vanish on $V \cap \mathbb{R}^d$. Therefore, if $(g_1, \ldots, g_m)$ is a basis of the real vector space of all real polynomials of degree at most D vanishing on $V \cap \mathbb{R}^d$, then the $g_1, \ldots, g_m$, regarded as complex polynomials, generate the complex vector space of all complex polynomials of degree at most D vanishing on V. It follows that the Hilbert function of the complex variety V is at least as large as the Hilbert function of the real variety $V \cap \mathbb{R}^d$. □

Lemma 1.8 (A generalized Bézout inequality). *Let $V \subseteq \mathbb{C}^d$ be an irreducible variety, let $f \in \mathbb{C}[x_1, \ldots, x_d]$ be a polynomial that does not vanish identically on V, and let $W_1, \ldots, W_k$ be the irreducible components of $V \cap Z_{\mathbb{C}}(f)$. Then all of the W_i have dimension* $\dim(V) - 1$*, and their degrees satisfy*

$$\sum_{i=1}^{k} \deg W_i \leq \deg(V) \deg(f).$$

Proof. We may assume that f is irreducible (if not, we decompose it into irreducible factors, use the lemma for each factor separately, and add up the degrees).

The first part about dimension of every irreducible component is exactly [Har77, Exercise I.1.8] (also see [Har77, Prop. I.7.1]).

As for the statement with degrees, we let $\overline{V} \subseteq \mathrm{P}\mathbb{C}^d$ be the projective closure of V, and similarly for $\overline{Z_{\mathbb{C}}(f)}$. Let $Y_1, \ldots, Y_m$ be the irreducible components of $\overline{V} \cap \overline{Z_{\mathbb{C}}(f)}$. By [Har77, Thm. I.7.7], we have $\sum_{i=1}^{m} \deg Y_i \leq \deg(\overline{V}) \deg(\overline{Z_{\mathbb{C}}(f)}) = \deg(V) \deg(f)$. For every W_i, the projective closure $\overline{W}_i$ is irreducible, and so it equals a unique $Y_{j(i)}$, and $\deg W_i \leq \deg Y_{j(i)}$. The lemma follows. Also see [Hei83, Thm. 1] for a similar statement. □

We will need to apply the lemma to a variety that is not necessarily irreducible. We will use that the degree is additive in the following sense: if $V_1, \ldots, V_k$ are the irreducible components of a variety V, with $\dim V_i = \dim V$ for all i, then $\deg V = \sum_{i=1}^{k} \deg V_i$.

We also need the property that a variety of degree Δ can be defined by polynomials of degree at most Δ.

Theorem 1.9 (Prop. 3 in [Hei83]). *Let V be an irreducible affine variety in $\mathbb{C}^d$. Then there exist $d+1$ polynomials $f_1, \ldots, f_{d+1} \in \mathbb{C}[x_1, \ldots, x_d]$ of degree at most* $\deg V$ *such that $V = Z_{\mathbb{C}}(f_1, \ldots, f_{d+1})$.*

Most of the time we will work with affine varieties. The only exception when we need projective ones is in Section 1.10.

1.3 Polynomial Partitions in Detail

Since the polynomial ham-sandwich theorem is central for polynomial partitions, we will explain it in detail. From that, we derive Theorems 1.1 and 1.2. We follow the exposition in [KMS12].

We say that a hyperplane h in $\mathbb{R}^d$ *bisects* a finite set $A \subset \mathbb{R}^d$ if both of the open subspaces of $\mathbb{R}^d$ bounded by h contain at most $|A|/2$ points of A. The discrete version of the standard *ham-sandwich theorem* ([ST42]) can be stated as follows: Every d finite sets $A_1, \dots, A_d \subset \mathbb{R}^d$ can be simultaneously bisected by a hyperplane.

We can generalize the notion of bisection to arbitrary polynomials: A polynomial $f \in \mathbb{R}[x_1, \dots, x_d]$ *bisects* a finite set $A \subset \mathbb{R}^d$ if each of the two sets $\{x \in \mathbb{R}^d\colon f(x) > 0\}$, $\{x \in \mathbb{R}^d\colon f(x) < 0\}$ contains at most $|A|/2$ points of A.

Now it is easy to derive the *polynomial ham-sandwich theorem* of Stone and Tukey [ST42].

Theorem 1.10 (Polynomial ham-sandwich, [ST42]). *Let $A_1, \dots, A_s \subset \mathbb{R}^d$ be finite sets, let D be an integer such that $\binom{D+d}{d} \geq s+1$. Then there exists a nonzero polynomial $f \in \mathbb{R}[x_1, \dots, x_d]$ of degree at most D that simultaneously bisects all the sets A_i, $1 \leq i \leq s$.*

Proof. First we note that the number of monomials[2] in d variables of degree at most D is $\binom{D+d}{d}$. In other words, $\binom{D+d}{d}$ is the number of d-tuples $(i_1, \dots, i_d)$ of nonnegative integers with $i_1 + \dots + i_d \leq D$.

We set $k := \binom{D+d}{d} - 1$. Let $J = \{(i_1, \dots, i_d)\colon 1 \leq i_1 + \dots + i_d \leq D, \text{ for } i_1, \dots, i_d \in \mathbb{Z}_0^+\}$; clearly $|J| = k$. Let $\Psi\colon \mathbb{R}^d \to \mathbb{R}^k$ denote the *Veronese map* given by

$$(x_1, \dots, x_d) \mapsto (x_1^{j_1} x_2^{j_2} \cdots x_d^{j_d})_{j=(j_1,\dots,j_d)\in J} \in \mathbb{R}^k.$$

We note that we can think of coordinates in $\mathbb{R}^k$ as indexed by d-tuples from J, in other words, every coordinate in $\mathbb{R}^k$ corresponds to exactly one nonzero monomial of degree at most D.

Let $A_i' := \Psi(A_i)$, for every $1 \leq i \leq s$. Since $s \leq k$, there is, by ham-sandwich theorem, a hyperplane H simultaneously bisecting $A_1', \dots, A_s'$. Consider the polynomial $f = h \circ \Psi$, where $h = 0$ is a linear equation of H. It follows that f belongs to $\mathbb{R}[x_1, \dots, x_d]$, degree of f is at most D and f simultaneously bisects each of the sets $A_1, \dots, A_s$. Indeed, let H have an equation $h_0 + \sum_{j \in J} h_j y_j$, where $h_j \in \mathbb{R}$ are some constants and y_j denotes the coordinates of $\mathbb{R}^k$. Since $f = h \circ \Psi = h_0 + \sum_{j\in J} h_j x_1^{j_1} \dots x_d^{j_d}$, it follows that $f \in \mathbb{R}[x_1, \dots, x_d]$ is of degree at most D. In order to show that f bisects each A_i, consider a point $a \in \mathbb{R}^d$ and let $a' = \Psi(a)$. Since $f(a) = h_0 + \langle g, a' \rangle$, where g is a vector with coordinates $h_1, \dots, h_k$ and $\langle \cdot, \cdot \rangle$ denotes the scalar product, and since h bisects every $A_i' = \Psi(A_i)$, the claim follows. □

Remark 1.11. *Using the bound $\binom{n}{k} \geq \left(\frac{n}{k}\right)^k$, we see that the degree D of the bisecting polynomial of s finite sets can be chosen as $ds^{1/d}$. Indeed, $\binom{D+d}{d} \geq \left(\frac{D+d}{d}\right)^d = (s^{1/d} + 1)^d \geq s+1$, hence D satisfies the requirement of the polynomial ham-sandwich theorem.*

Sketch of the proof of Theorem 1.1. In order to find a $\frac{1}{r}$-partitioning polynomial for P of cardinality n, it is enough to construct a "suitable" sequence of polynomial ham-sandwich cuts, which partition the given point set P into a specified number t of subsets, each consisting of at most cn/t points, where $c > 1$ is a constant. This technique first bisects the original point set P into two halves, using a polynomial f_1. It then bisects each of these two sets into two halves, using a second polynomial f_2, bisects each of the four resulting subsets using a third polynomial f_3, and so on, until the desired number t of subsets is obtained (or exceeded). Set $t := cr$. Then clearly every subset contains at most $cn/t = n/r$ points. The product $f = f_1 f_2 f_3 \cdots$ of these bisecting polynomials is the desired $\frac{1}{r}$-*partitioning polynomial*, and, using Remark 1.11, it can be shown its degree is $D = O(t^{1/d}) = O(r^{1/d})$. □

[2]By *monomial* we mean $x_1^{i_1} \dots, x_d^{i_d}$, where $i_1, \dots, i_d$ are nonnegative integers.

Note that the resulting partition is not exhaustive, as some points of P may lie in the zero set $Z(f)$ of f. Note that in general it makes sense to take $t \leq n$. If $t > n$ we can, following the technique used in [EKS11, GK10], find a polynomial f of degree $O(n^{1/d}) = O(t^{1/d})$ that vanishes at all the points of P. In this case all the subsets in the resulting partition of P are empty, except for $P \cap Z(f) = P$.

1.3.1 A second partitioning polynomial

Using a variant of polynomial ham-sandwich theorem, we prove Theorem 1.2.

Proof of Theorem 1.2. Let us denote the $\frac{1}{r}$-partitioning polynomial we are looking for by g. As in the standard polynomial partitioning technique, we obtain g as the product of logarithmically many bisecting polynomials, each obtained by applying a variant of the polynomial ham-sandwich theorem to a current collection of subsets of P. The difference, though, is that we want to ensure that each of the bisecting polynomials is not divisible by f; since f is irreducible, this ensures coprimality of g with f. Reviewing the construction of polynomial ham-sandwich cuts, as outlined above, we see that all that is needed is to come up with some sufficiently large finite set of monomials, of an appropriate maximum degree, so that no nontrivial linear combination of these monomials can be divisible by f. We then use a restriction of the Veronese map defined by this subset of monomials, and the standard ham-sandwich theorem in the resulting high-dimensional space, to obtain the desired polynomial.

Let $x_1^{i_1} x_2^{i_2} \cdots x_d^{i_d}$ be the *leading term* of f, in the sense that $i_1 + \cdots + i_d = D$ and $(i_1, \ldots, i_d)$ is largest in the lexicographical order among all the d-tuples of exponents of the monomials of f of degree D. Let s be the desired number of sets that we want to bisect by a single partitioning polynomial. For that we need a collection of s monomials whose degrees are not too large and which span only polynomials not divisible by f. If, say, $s < (D/d)^d$, then we can use all monomials $x_1^{j_1} \cdots x_d^{j_d}$ such that $j_k \leq s^{1/d} < D/d$, for $k = 1, \ldots, d$. Clearly, any nontrivial linear combination of these monomials cannot be divisible by f. In this case the degree of the resulting partitioning polynomial is $O(s^{1/d})$. If $s \geq (D/d)^d$, then we consider the set of all monomials $x_1^{j_1} \cdots x_d^{j_d}$ such that $j_k < i_k$ for at least one $k = 1, \ldots, d$, and $\max\{j_1, \ldots, j_d\} \leq \hat{D}$ for a suitable integer $\hat{D}$, which we specify below (the actual degree of the bisecting polynomial under construction will then be at most $d\hat{D}$). Any nontrivial polynomial h which is a linear combination of these monomials cannot be divisible by f. Indeed, if $h = fh_1$ for some polynomial h_1 then the product of the leading terms of f and of h_1 cannot be canceled out by the other monomials of the product, and, by construction, h cannot contain this monomial. The number of monomials in this set is $\Theta(i_1 \hat{D}^{d-1} + i_2 \hat{D}^{d-1} + \cdots + i_d \hat{D}^{d-1}) = \Theta(D\hat{D}^{d-1})$. We thus pick $\hat{D} = \Theta((s/D)^{1/(d-1)})$ so that we indeed get s monomials. As noted above, the degree of the resulting bisecting polynomial in this case is $O((s/D)^{1/(d-1)})$.

Set $t := \lceil \log_2 r \rceil$. We now proceed to construct the required partitioning of P into about r sets, by a sequence of t polynomials $g_0, g_1, \ldots, g_{t-1}$ where g_j bisects 2^j subsets of P, each of size at most $|P|/2^j$. For every j such that $s = 2^j < (D/d)^d$ we construct, as shown above, a polynomial of degree $O(s^{1/d}) = O(2^{j/d})$. For the indices j with $s = 2^j \geq (D/d)^d$ we construct a polynomial of degree $O((s/D)^{1/(d-1)}) = O((2^j/D)^{1/(d-1)})$. Set $g := g_0 g_1 \cdots g_{t-1}$. By construction, g is an $\frac{1}{r}$-partitioning polynomial for P, since

$|P|/2^t \leq |P|/r$. Moreover, f does not divide any g_j, $j = 0, \ldots, t-1$, so g is co-prime with f. It remains to bound the degree of g:

$$\begin{aligned}\deg g &= \sum_{i=0}^{t-1} \deg g_i \leq \sum_{j<d\log_2(D/d)} O(2^{j/d}) + \sum_{j\geq d\log_2(D/d)}^{t-1} O\left(\left(\frac{2^j}{D}\right)^{1/(d-1)}\right)\\ &= O\left(2^{\log_2(D/d)}\right) + O\left(\left(\frac{2^t}{D}\right)^{1/(d-1)}\right) = O\left(D + \left(\frac{r}{D}\right)^{1/(d-1)}\right),\end{aligned}$$

The proof is complete. □

1.3.2 On a third partitioning polynomial – how to continue?

As we have seen, it may happen that many points of P lie in $Z(f,g)$. Let us assume that $Z(f,g) \subset \mathbb{R}^d$ is an irreducible variety of dimension $d-2$. The natural idea is, similarly as in the previous case, to find a further partitioning polynomial h such that $\dim(Z(f,g) \cap Z(h)) \leq d-3$. In this section we describe some limitations of such approach.

Let us first describe a condition on h under which $\dim(Z(f,g) \cap Z(h)) \leq d-3$. We claim, more generally, that if $V \subset \mathbb{R}^d$ is an irreducible variety of dimension k, then $\dim(V \cap Z(h)) \leq k-1$ if and only if $h \notin \mathbf{I}(V)$, where $\mathbf{I}(V)$ is the ideal of all polynomials vanishing on V. Indeed, the implication $\Rightarrow$ is trivial. To show the second one, let $W := V \cap Z(h)$ and $J := \langle \mathbf{I}(V), h\rangle$. Note that Since $h \notin \mathbf{I}(V)$, we have $\mathbf{I}(V) \subsetneq J \subseteq \mathbf{I}(W)$. Because V is irreducible, $\mathbf{I}(V)$ is a prime ideal and the inequality $\dim W < \dim V$ follows from the Krull dimension definition[3].

From that we can easily derive one claim mentioned in the introduction, more precisely, given an irreducible polynomial $f \in \mathbb{R}[x_1, \ldots, x_d]$ and a polynomial g coprime with f, then $\dim Z(f,g) \leq d-2$. Without loss of generality we can assume that $\dim Z(f) = d-1$, otherwise the inequality holds trivially. By [BCR98, Thm. 4.5.1], $\dim Z(f) = d-1$ if and only if $\mathbf{I}(Z(f)) = \langle f \rangle$. Hence, the condition on coprimality of g with f implies that $g \notin \langle f \rangle = \mathbf{I}(Z(f))$ and the inequality follows from the previous claim.

We would like to find a third partitioning polynomial h for a finite point set $Q \subset Z(f,g)$ such that $\dim Z(f,g,h) \leq d-3$. Recall that $Z(f,g)$, denoted by V, is an irreducible variety of dimension $d-2$. We may assume that f and g are coprime. The first idea might be to define h as f^2+g^2. Since $Z(h) = Z(f,g)$ and the polynomial h is irreducible, we can apply the second polynomial partitioning method, described in the previous section, on h. However, Theorem 1.2 provides a polynomial h' for which only $\dim Z(h,h') \leq d-2$ is guarantied. Moreover, the equality can be easily achieved. Indeed, it can happen that $h' = f$. Then h' is coprime with h and $Z(h,h') = Z(f,g)$. The equality follows.

The second idea of constructing h follows from the analysis of the second polynomial partitioning method. A closer look at the proof of Theorem 1.2 shows that in order to get a partitioning polynomial of degree $O(\hat{D})$ we need a collection of $\Theta(D\hat{D}^{d-1})$ "suitable" monomials of degree $O(\hat{D})$, where $D = \deg f$ and $\hat{D} \geq D$. From that we can get an impression that for finding a third partitioning polynomial h of degree $O(\hat{D})$ it is enough to find a "suitable" collection of $\Theta(DE\hat{D}^{d-2})$ monomials of degree $O(\hat{D})$,

[3] We use the fact that the preimage of a prime ideal under a ring homomorphism is a prime ideal.

where $D = \deg f$, $E = \deg g$ and $\hat{D} \geq E$. Here we need to specify which collections of monomials are suitable. By the discussion above, $\dim(V \cap Z(h)) \leq d-3$ if and only if $h \notin \mathbf{I}(V)$. We say that a collection $\mathcal{C}$ of monomials is *suitable* for V if no nontrivial linear combination of monomials from $\mathcal{C}$ belongs to $\mathbf{I}(V)$.

However, as shown in Example 1.12, there are irreducible varieties for which there is no suitable collection of $\Theta(DE\hat{D}^{d-2})$ monomials of degree $O(\hat{D})$.

Let us assume, for simplicity, that $\deg f = \deg g$.

Example 1.12. *Let $d \geq 3, D \geq 1$ and $\hat{D} \geq D$ be integers. Let $f = y - x^D, g = z - x^D$ be polynomials in $\mathbb{R}[x, y, z, x_4, x_5, \ldots, x_d]$. Then $V := Z(f, g)$ is an irreducible variety of dimension $d-2$ and any collection of monomials of degrees at most $\hat{D}$ which is suitable for V has size $O(D\hat{D}^{d-2})$.*

Observe that V can be parametrized by $\{(t, t^D, t^D, u_1, u_2, \ldots, u_{d-3}) \colon t, u_i \in \mathbb{R}\}$ and $\dim V = d-2$. By [CLO07, Prop. 4.5.5], every variety, which can be parametrized, is irreducible. It can be also checked that $\langle f, g \rangle = \mathbf{I}(V)$. For the sake of completeness, we will prove it later. From the parametrization of V, it follows that $h \in \mathbf{I}(V)$ if and only if $h(t, t^D, t^D, u_1, \ldots, u_{d-3}) = 0$.

First, let us assume that $d = 3$ and let $\mathcal{C}$ be a suitable collection for V consisting of monomials of degree at most $\hat{D}$.

To each monomial $x^i y^j z^k$ we assign t^ℓ, where $\ell = i + (j+k)D$, simply by substituting $x = t$, $y = t^D$, $z = t^D$. We claim that for each t^ℓ which was associated with some monomial $x^i y^j z^k$, at most one of these corresponding monomials is contained in $\mathcal{C}$. Indeed, it is clear that if two distinct monomials correspond to the same t^ℓ, then their difference vanishes on V and hence belongs to $\mathbf{I}(V)$. The size of $\mathcal{C}$ is clearly at most $D\hat{D}$ since there is at most $D\hat{D}$ different values of $i + (j+k)D$, where $i + j + k \leq \hat{D}$ and $D \leq \hat{D}$.

The generalization for $d \geq 4$ is straightforward. Indeed, we associate each monomial $x^i y^j z^k x_1^{\ell_1} \cdots x_{d-3}^{\ell_{d-3}}$ with $t^\ell x_1^{\ell_1} \cdots x_{d-3}^{\ell_{d-3}}$, where $\ell = i + (j+k)D$ simply by substituting $x = t, y = t^D, z = t^D$. Then any suitable collection $\mathcal{C}$ of monomials of degrees at most $\hat{D}$ has at most one monomial corresponding to a fixed $t^\ell x_1^{\ell_1} \cdots x_{d-3}^{\ell_{d-3}}$. It follows that the size of $\mathcal{C}$ is at most $O(D\hat{D}^{d-2})$.

It remains to show that $I = \mathbf{I}(V)$ where $I := \langle f, g \rangle$. The method is standard and can be found, e.g., in [CLO07].

Let us assume, for convenience, that $x_1 = x, x_2 = y$ and $x_3 = z$. We first observe that each polynomial $h \in \mathbb{R}[x_1, \ldots, x_d]$ can be written in the form

$$h = h_1(x_2 - x_1^D) + h_2(x_3 - x_1^D) + r,$$

where $r \in \mathbb{R}[x_1, x_4, \ldots, x_d]$ and $h_1, h_2 \in \mathbb{R}[x_1, \ldots, x_d]$.

Indeed, this is true for monomial $x_1^{i_1} \cdots x_d^{i_d}$; according to the binomial theorem we have:

$$x_1^{i_1} \cdots x_d^{i_d} = x_1^{i_1}(x_1^D + (x_2 - x_1^D))^{i_2}(x_1^D - (x_3 - x_1^D))^{i_3} x_4^{i_4} \cdots x_d^{i_d}$$
$$= x_1^{i_1}(x_1^{i_2 D} + \text{terms involving}(x_2 - x_1^D))(x_1^{i_3 D} + \text{terms involving}(x_3 - x_1^D)) x_4^{i_4} \cdots x_d^{i_d}.$$

Multiplying this out shows that

$$x_1^{i_1} \cdots x_d^{i_d} = h_1(x_2 - x_1^D) + h_2(x_3 - x_1^D) + x_1^{i_1 + (i_2 + i_3)D} x_4^{i_4} \cdots x_d^{i_d},$$

for some $h_1, h_2 \in \mathbb{R}[x_1, \ldots, x_d]$. Since an arbitrary $h \in \mathbb{R}[x_1, \ldots, x_d]$ is an $\mathbb{R}$-linear combination of monomials, the decomposition follows.

Because the inclusion $I \subseteq \mathbf{I}(V)$ is clear, it remains to show $\mathbf{I}(V) \subseteq I$. Let $h \in \mathbf{I}(V)$ and let $h = h_1(x_2 - x_1^D) + h_2(x_3 - x_1^D) + r$. To prove that r is zero polynomial we use the parametrization $(t, t^D, t^D, u_1, u_2, \ldots, u_{d-3})$ of V. Since h vanishes on V, we get

$$0 = h(t, t^D, t^D, u_1, u_2, \ldots, u_{d-3}) = r(t, u_1, u_2, \ldots, u_{d-3}). \tag{1.1}$$

In other words, the corresponding function $r\colon \mathbb{R}^{d-2} \to \mathbb{R}$ is the zero function, hence the polynomial $r \in \mathbb{R}[x_1, x_4, \ldots, x_d]$ is zero polynomial[CLO07, Prop. 1.1.5] and the claim follows. □

It turns out that what we actually want is to partition the variety $Z(f, g)$ and hence it seems reasonable that the degree of partitioning polynomial should depend on some property of the variety rather than on the degrees of its defining polynomials. We note that one variety can be defined by different sets of polynomials with very different degrees. Indeed, set $f = y - x^{10}$, $g = z - x^{100}$. It can be easily verified that $Z(f, g)$ can be parametrized as $\{(t, t^{10}, t^{100})\colon t \in \mathbb{R}\}$. However, if we set $f' = y - x^{10}$, $g' = z - y^{10}$, then $Z(f, g) = Z(f', g')$.

In Section 1.10 we show that for a complex irreducible variety its degree is indeed the right parameter. More precisely, we show that for a complex irreducible variety V of (complex) dimension k and a finite point set $Q \subset \mathbb{R}^d \cap V$ there exists a real $(1/r)$-partitioning polynomial for Q of degree at most $O(\Delta + (r/\Delta)^{1/k})$, where Δ is the degree of V (Lemma 1.14). However, now we concentrate on the proof of Theorem 1.3 where we do not need an optimal bound for the degree of the partitioning polynomial.

1.4 A key lemma: Partitioning polynomial that does not vanish on a variety

In this section we establish the following lemma, which is crucial for the proof of Theorem 1.3 and which will allow us to deal with the exceptional sets and iterate the construction of a partitioning polynomial. Although we are dealing with a problem in $\mathbb{R}^d$, it will be more convenient to work with complex varieties. This is because algebraic varieties over an algebraically closed field have some nice properties that fail for real varieties in general.

Lemma 1.13 (Key lemma)**.** *Let $V \subseteq \mathbb{C}^d$ be a complex algebraic variety of dimension $k \geq 1$, such that all of its irreducible components V_j have dimension k as well. Let $Q \subset V \cap \mathbb{R}^d$ be a finite point set, and let $r > 1$ be a parameter. Then there exists a real $\frac{1}{r}$-partitioning polynomial g for Q of degree at most $D = O(r^{1/k})$ that does not vanish identically on any of the irreducible components V_j of V.*

Assuming, moreover, that V is irreducible, we prove the following strengthening of the key lemma:

Lemma 1.14. *Let $V \subseteq \mathbb{C}^d$ be an irreducible complex algebraic variety of dimension $k \geq 1$ and degree Δ. Let $Q \subset V \cap \mathbb{R}^d$ be a finite point set, and let $r > 1$ be a parameter. Then there exists a real $\frac{1}{r}$-partitioning polynomial g for Q of degree at most $O\left(\Delta + \left(\frac{r}{\Delta}\right)^{1/k}\right)$ that does not vanish identically on V.*

We note that for $r \geq \Delta^{k+1}$, the bound reduces to $O\left(\left(\frac{r}{\Delta}\right)^{1/k}\right)$ and hence it resolves a conjecture of Matoušek and the author mentioned in [MP15, Conj. 3.2]. We also note that Basu and Sombra propose a conjecture similar to our lemma, see [BS14, Conj. 3.4].

We remark that we cannot use Lemma 1.14 for our range searching application unless we know how to effectively decompose a variety into irreducibles.

Before proving the key lemma, we first sketch the idea. The proof is based on a projection trick. Let us consider the *standard projection* $\pi_d\colon \mathbb{C}^d \to \mathbb{C}^{d-1}$ given by $(a_1, \ldots, a_d) \mapsto (a_1, \ldots, a_{d-1})$, i.e., forgetting the last coordinate. The standard projection of an affine variety need not be a variety in general (consider, e.g., the projection of the hyperbola $Z(xy-1)$ on the x-axis). However, for every variety of dimension at most $d-1$, there is a simple linear change of coordinates in $\mathbb{C}^d$ (Lemma 1.16) after which the image of V under the standard projection is a variety in $\mathbb{C}^{d-1}$ (Theorem 1.15). Moreover, this projection preserves the dimension of the variety (Theorem 1.15).

The idea of the proof of the key lemma is to project the given k-dimensional complex variety V onto $\mathbb{C}^k$, by iterating the standard projection, and, if necessary, coordinate changes in such a way that the image of V is all of $\mathbb{C}^k$ (Corollary 1.17). Then we find a $\frac{1}{r}$-partitioning polynomial for the projection of the given point set Q by the Guth–Katz method, and we pull it back to a $\frac{1}{r}$-partitioning polynomial in $\mathbb{R}^d$.

We now present this approach in more detail. We begin with a well-known sufficient condition guaranteeing that the standard projection of a variety is a variety of the same dimension.

Theorem 1.15 (Projection theorem)**.** *Let $I \subset \mathbb{C}[x_1, \ldots, x_d]$ be an ideal, $d \geq 2$, and let $J := I \cap \mathbb{C}[x_1, \ldots, x_{d-1}]$ be the ideal consisting of all polynomials in I that do not contain the variable x_d. Suppose that I contains a nonconstant polynomial f, with $D = \deg f \geq 1$, in which the monomial x_d^D appears with a nonzero coefficient. Let $V = V(I)$ be a complex variety defined as the zero locus of all polynomials in I. Then the image $\pi_d(V)$ under the standard projection $\pi_d\colon \mathbb{C}^d \to \mathbb{C}^{d-1}$ is the variety $Z_{\mathbb{C}}(J) \subseteq \mathbb{C}^{d-1}$, and $\dim \pi_d(V) = \dim V$. Moreover, each point of $\mathbb{C}^{d-1}$ has a finite number of preimages contained in V.*

Proof. Theorem 1.68 in [DP13] contains everything in the theorem except for the claims $\dim \pi_d(V) = \dim V$ and that V contains a finite number of preimages of each point in $\mathbb{C}^{d-1}$. For these claims, which are also standard, we first observe that, for every point $a \in \pi_d(V)$, the x_d-coordinates of these preimages are roots of the nonzero univariate polynomial $f_a(x_d) := f(a_1, \ldots, a_{d-1}, x_d)$. (Since every univariate polynomial has finite number of roots, the second claim follows.) In other words the extension $\mathbb{C}[x_1, \ldots, x_{d-1}]/J \subseteq \mathbb{C}[x_1, \ldots, x_d]/I$ is integral.[4] By [HS06, Thm. 2.2.5], integral extension preserves (Krull) dimension. □

The next standard lemma (a simple form of the *Noether normalization* for infinite fields) implies that the condition in the projection theorem can always be achieved by a suitable change of coordinates. See, e.g., [DP13, Lemma 1.69].

Lemma 1.16. *Let $f \in \mathbb{C}[x_1, \ldots, x_d]$ be a polynomial of degree $D \geq 1$. Then there are coefficients $\lambda_1, \ldots, \lambda_{d-1}$ such that*

$$f'(x_1, \ldots, x_d) := f(x_1 + \lambda_1 x_d, \ldots, x_{d-1} + \lambda_{d-1} x_d, x_d)$$

[4]A ring S is an *integral extension* of a subring $R \subseteq S$ if all elements of S are roots of monic polynomials in $R[x]$.

is a polynomial of degree D in which the monomial x_d^D has a nonzero coefficient. This holds for a generic choice of the λ_i, meaning that there is a nonzero polynomial $g \in \mathbb{C}[y_1, \dots, y_{d-1}]$ such that f' satisfies the condition above whenever $g(\lambda_1, \dots, \lambda_{d-1}) \neq 0$. Consequently, the condition on f' holds for almost all choices of a real *vector $(\lambda_1, \dots, \lambda_{d-1})$.*

By combining the projection theorem with Lemma 1.16 and iterating, we obtain the following consequence:

Corollary 1.17. *Let $V \subset \mathbb{C}^d$ be a complex variety of dimension k, $1 \leq k \leq d-1$, for which all irreducible components also have dimension k, and let $Q \subset V \cap \mathbb{R}^d$ be a finite set of points. Then there is a linear map $\pi \colon \mathbb{C}^d \to \mathbb{C}^k$, whose matrix w.r.t. the standard bases is real, such that $\pi(V_j) = \mathbb{C}^k$ for every irreducible component V_j of V and $|\pi(Q)| = |Q|$.*

Proof. First, we define an auxiliary variety $W := V \cup \bigcup_{q_i, q_j \in Q} \overleftrightarrow{q_i q_j}$, where $\overleftrightarrow{q_i q_j}$ denotes the line spanned by two different points q_i, q_j. Note that $\dim W = \dim V = k$. We construct π iteratively by composing standard projections and appropriate coordinate changes. First we choose a nonzero polynomial f vanishing on W, and we fix a change of coordinates as in Lemma 1.16 so that the corresponding polynomial f' is as in the projection theorem. Letting $\pi'_d \colon \mathbb{C}^d \to \mathbb{C}^{d-1}$ be the composition of the standard projection π_d with this coordinate change, we get that $\pi'_d(W)$ is a variety and $\dim \pi'_d(W) = k$. Clearly, $|\pi'_d(Q)| = |Q|$, since by the projection theorem every point in $\pi'_d(W)$ has a finite number of preimages in W.

Let V_j be an irreducible component of V. Then f vanishes on V_j as well, and applying the projection theorem with V_j instead of W, we get that $\pi'_d(V_j)$ is a k-dimensional variety in $\mathbb{C}^{d-1}$ as well.

We define $\pi'_i \colon \mathbb{C}^i \to \mathbb{C}^{i-1}$, $i = d-1, d-2, \dots, k+1$, analogously; to get π'_i, we use some nonzero polynomial f that vanishes on the k-dimensional variety $\pi'_{i+1} \circ \cdots \circ \pi'_d(W)$. The desired projection π is the composition $\pi := \pi'_{k+1} \circ \cdots \circ \pi'_d$. Consequently, $|\pi(Q)| = |Q|$.

We get that $\pi(W)$ is a k-dimensional variety in $\mathbb{C}^k$, and so is $\pi(V_j)$ for every irreducible component V_j of V. But the only k-dimensional variety in $\mathbb{C}^k$ is $\mathbb{C}^k$, and the corollary follows. $\square$

Now we are ready to prove the key lemma.

Proof of Lemma 1.13. Given the k-dimensional complex variety V and the n-point set $Q \subset \mathbb{R}^d$ as in the key lemma, we consider a projection $\pi \colon \mathbb{C}^d \to \mathbb{C}^k$ as in Corollary 1.17.

Since the matrix of π is real, we can regard $Q' := \pi(Q)$ as a subset of $\mathbb{R}^k$.

We apply the original Guth–Katz polynomial partition theorem to Q', which yields a $\frac{1}{r}$-partitioning polynomial $g' \in \mathbb{R}[y_1, \dots, y_k]$ for Q' of degree $D = O(r^{1/k})$.

We define a polynomial $g \in \mathbb{R}[x_1, \dots, x_d]$ as the pullback of g', i.e., $g(x) := g'(\pi(x))$. We have $\deg g = \deg g'$ since π is linear and surjective.

Moreover, g is a $\frac{1}{r}$-partitioning polynomial for Q, since if $\pi(q)$ and $\pi(q')$ lie in different components of $\mathbb{R}^k \setminus Z(g')$, then q and q' lie in different components of $\mathbb{R}^d \setminus Z(g)$ (indeed, if not, a path γ connecting q to q' and avoiding $Z(g)$ would project to a path γ' connecting $\pi(q)$ to $\pi(q')$ and avoiding $Z(g')$).

Finally, since g' does not vanish identically on $\mathbb{C}^k$ and $\pi(V_j) = \mathbb{C}^k$ for every j, the polynomial g does not vanish identically on any of the irreducible components V_j. The key lemma is proved. $\square$

In order to prove Theorem 1.4 we will need the following variant of the key lemma for $k = 1$.

Lemma 1.18. *Let us assume the same as in the key lemma (Lemma 1.13) and let $k = 1$. Then, if $r \le |Q|$ we can find a real $\frac{1}{r}$-partitioning polynomial g for Q of degree $O(r)$ such that the exceptional set $Q^* = Q \cap Z(g)$ is empty.*

Proof. We proceed as in the proof of the key lemma. Using Corollary 1.17, we can regard Q' as a subset of $\mathbb{R}$. Clearly, we can divide $\mathbb{R}$ into about r connected parts such that each part contains at most Q'/r points of Q'. Since $r \le |Q| = |Q'|$, we can also assume that all points of Q' are contained in the interior of the corresponding connected part. The "border" points of the partition define a zero set of a $\frac{1}{r}$-partitioning polynomial for Q'. Saying differently, we have found a $\frac{1}{r}$-partitioning polynomial g' for Q' of degree $O(r)$ such that the exceptional set $Q'^* = Q^* \cap Z(g')$ is empty. The rest is clear; we continue exactly as in the proof of the key lemma. □

1.5 Algebraic preliminaries II

Throughout the thesis we assume that we are working in the *Real RAM* model of computation where arithmetic operations with arbitrary real numbers can be performed exactly and in unit time. This is the most usual model in computational geometry.

We could also consider the bit model (a.k.a. Turing machine model), assuming the input points rational or, say, algebraic. Then the analysis would be more complicated, but we believe that, with sufficient care, bounds analogous to those we obtain in the Real RAM model can be derived as well, with an extra multiplicative term polynomial in the bit size of the input numbers. For example, the algorithms of real algebraic geometry we use are also analyzed in the bit model in [BPR03], and the polynomiality claims we rely on still hold. However, at present we do not consider this issue sufficiently important to warrant the additional complication of the thesis.

1.5.1 Ideals and Gröbner bases

For polynomials $f_1, \dots, f_m \in \mathbb{C}[x_1, \dots, x_d]$, the *ideal* I generated by $f_1, \dots, f_m$ is the set of all polynomials of the form $h_1 f_1 + \cdots + h_m f_m$, $h_1, \dots, h_m \in \mathbb{C}[x_1, \dots, x_d]$. Every such ideal has a *Gröbner basis*, which is a set of polynomials that also generates I and has certain favorable properties; see, e.g., [CLO07] for an introduction.

Each Gröbner basis is associated with a certain *monomial ordering*. We will use only Gröbner bases with respect to a *lexicographic ordering*, where monomials in the variables $x_1, \dots, x_d$ are first ordered according to the powers of x_d, then those with the same power of x_d are ordered according to powers of x_{d-1}, etc. In other words, we consider lexicographic ordering w.r.t. $x_d > x_{d-1} > \cdots > x_1$.

We will need the following theorem:

Theorem 1.19. *Assuming d fixed and given polynomials $f_1, \dots, f_m \in \mathbb{C}[x_1, \dots, x_d]$ with $\deg f_i \ge 1$, a Gröbner basis of the ideal generated by the f_i can be computed in time polynomial in $\sum_{i=1}^{m} \deg f_i$.*

We have not found an explicit reference in the literature that would provide Theorem 1.19. In particular, for the usual Buchberger algorithm and variations of it, only much worse bounds seem to be known. However, Theorem 1.19 follows by inspecting the method of Kühnle and Mayr [KM96] for finding a Gröbner basis in exponential space. (Also see [MR11] for a newer algorithm.)

Before providing the details, we need one definition: For any polynomial $h \in \mathbb{C}[x_1, \dots, x_d]$, the *normal form* $\mathsf{NF}(h)$ w.r.t. $I \subseteq \mathbb{C}[x_1, \dots, x_d]$ is the unique *irreducible*[5] polynomial w.r.t. I in the coset[6] $h + I$. Recall that we have fixed lexicographic ordering.[7] We note that Kühnle and Mayr work over the field $\mathbb{Q}$, however, the theoretical background works also for $\mathbb{C}$. Let $I \subseteq \mathbb{C}[x_1, \dots, x_d]$ be an ideal whose Gröbner basis we want to compute and assume it is generated by m polynomials of degree bounded by D.

(i) First important lemma [KM96, Sect. 5],[MR11, Lemma 3] is that the *reduced* Gröbner basis is always equal to the set of all the polynomials $h - \mathsf{NF}(h)$, where h is a monomial *minimally reducible*[8] w.r.t. I.

(ii) Let $h \in \mathbb{C}[x_1, \dots, x_d]$ be arbitrary but fixed. Our next goal is to compute $\mathsf{NF}(h)$ w.r.t. I. Since $h - \mathsf{NF}(h) \in I$, there is a representation

$$h - \mathsf{NF}(h) = \sum_{i=1}^{m} c_i f_i \quad \text{with} \quad c_1, \dots, c_m \in \mathbb{C}[x_1, \dots, x_d]. \tag{1.2}$$

The next step is to rewrite the polynomial equation (1.2) to a system of linear equations. Recall that h and f_i's are fixed and $\mathsf{NF}(h)$ and c_i's are unknowns. Let us assume that $\deg c_i \leq E$ for all i and some E. Expanding all polynomials h, f_i, c_i and also the polynomial $r := \mathsf{NF}(h)$ to sums of monomials and comparing coefficients of left and right sides in (1.2), we get one linear equation for every term. If $\deg \mathsf{NF}(h) \leq N$ for some N, it can be shown that there are at most $(\max(N, D + E))^d$ equations in no more than $N^d + mE^d$ unknowns. It follows that all these linear equations can be rewritten into a single matrix equation and the size of the matrix is bounded by $N^d + m(D + E)^d$. For more details we refer to [KM96, Sect. 3]. Note that it can happen that there are more unknowns than equations. Fortunately, since we are interested in a solution with minimal r (w.r.t. lexicographic ordering), we can always decrease the number of unknowns by putting the coefficient corresponding to the largest monomial of r to be zero. For more details (and also example) we again refer to [KM96, Sect. 3].

(iii) Now we want to bound degrees of c_i's and also the degree of $\mathsf{NF}(h)$. By Hermann [Her26, MM82], the degrees of c_i's are bounded by $E := \deg(h - \mathsf{NF}(h)) + (mD)^{2^d}$. Dubé [Dub90] showed the existence of a Gröbner basis G for I where the degrees of all polynomials in G are bounded by $M := 2(D^2/2 + D)^{2^{d-1}}$. Using this bound,

[5]A polynomial h is *reducible* w.r.t. I if $\mathsf{supp}(h) \cap \langle \ell m(I) \rangle \neq \emptyset$, where the support of h is a set of all monomials occurring in h (i.e., having nonzero coefficient) and $\langle \ell m(I) \rangle = \langle \ell m(f) \colon f \in I \rangle$ is an ideal of all *leading monomials* of I, where leading monomial $\ell m(f)$ is the largest monomial occurring in f.

[6]$h + I = \{h + f \colon f \in I\}$.

[7]We note that the algorithm by [KM96] requires the monomial ordering given by rational weight matrix. The weight matrix of lexicographic ordering consists just of zero's and one's, and hence it is rational. See [KM96] for details.

[8]A monomial h is *minimally reducible* w.r.t. I if it is reducible w.r.t. I but none of its proper divisors is reducible w.r.t. I.

Kühnle and Mayr [KM96, Sect. 2] showed that the degree of the normal form of h w.r.t. I can be always bounded by $N := ((M+1)^d \deg(h))^{d+1}$.

(iv) It follows that to compute reduced Gröbner basis of I it is enough to enumerate all monomials up to Dubé's bound and calculate their normal forms and normal forms of all its direct divisors. This can be done by solving the system of linear equations described in (ii).

In order to turn the described method into an algorithm, we have to be able to efficiently solve a system of linear equations. Kühnle and Mayr used Turing machines, that is why they need to work over $\mathbb{Q}$. Since we work with the Real RAM model of computation which allows arithmetic operations with arbitrary real numbers (in unit time), we can use the described algorithm over $\mathbb{C}$ as well.

Now we are ready to prove Theorem 1.19.

Proof of Theorem 1.19. Clearly $D \leq \sum_{i=1}^{m} \deg f_i$ and $m \leq \sum_{i=1}^{m} \deg f_i$, since $\deg f_i \geq 1$ for every i. It follows from (i)–(iv) that, for d fixed, the Gröbner basis can be computed in time polynomial in $\sum_{i=1}^{m} \deg f_i$. Indeed, by (ii) and (iii), the normal form of a polynomial of degree bounded by $O(D)$ can be computed in time polynomial in D, and hence also in $\sum_{i=1}^{m} \deg f_i$. According to (iv), the step (ii) is repeated polynomially many times; the claim follows. □

1.5.2 Tools from real algebraic geometry

Let $\mathcal{F} \subset \mathbb{R}[x_1, \ldots, x_d]$ be a finite set of polynomials. The *arrangement* of (the zero sets of) $\mathcal{F}$ is the partition of $\mathbb{R}^d$ into maximal relatively open connected subsets, called *cells*, such that for each cell C there is a subset $\mathcal{F}_C \subseteq \mathcal{F}$ such that $C \subseteq Z(f)$ for all $f \in \mathcal{F}_C$ and $C \cap Z(f) = \emptyset$ for all $f \in \mathcal{F} \setminus \mathcal{F}_C$.

Similar to [AMS13], a crucial tool for us is the following theorem of Barone and Basu.

Theorem 1.20 (Barone and Basu [BB12]). *Let V be a k-dimensional algebraic variety in $\mathbb{R}^d$ defined by a finite set $\mathcal{F}$ of d-variate real polynomials, each of degree at most D, and let $\mathcal{G}$ be a set of s polynomials of degree at most $E \geq D$. Then the number of those cells of the arrangement of the zero sets of $\mathcal{F} \cup \mathcal{G}$ that are contained in V is bounded by $O(1)^d D^{d-k}(sE)^k$.*

We will be using the theorem only for constant d and $\mathcal{G} = \{g\}$ consisting of a single polynomial to get an upper bound of $O(D^{d-k}E^k)$ on the number of connected components of $V \setminus Z(g)$.

For the range searching algorithm, we also need the following algorithmic result on the construction of arrangements.

Theorem 1.21 (Basu, Pollack and Roy [BPR03, Thm. 16.18]). *Let $\mathcal{F} = \{f_1, \ldots, f_m\}$ be a set of m real d-variate polynomials, each of degree at most D. Then the arrangement of the zero sets of $\mathcal{F}$ in $\mathbb{R}^d$ has at most $(mD)^{O(d)}$ cells, and it can be computed in time at most $T = m^{d+1}D^{O(d^4)}$. Each cell is described as a semialgebraic set using at most T polynomials of degree bounded by $D^{O(d^3)}$. Moreover, the algorithm supplies adjacency information for the cells, indicating which cells are contained in the boundary of each cell, and it also supplies an explicitly given point in each cell.*

1.6 Proofs of multilevel partition theorems

Here we prove Theorems 1.3 and 1.4 and we start with the first one.

We use the key lemma to construct the multilevel partition in Theorem 1.3. Thus, we are given an n-point set $P \subset \mathbb{R}^d$ and a parameter $r > 1$.

We proceed in d steps. The parameters $r_1, r_2, \ldots, r_d$ are set as follows:

$$r_1 := r, \quad r_{i+1} := r_i^c, \quad i = 1, 2, \ldots, d-1,$$

where c is a sufficiently large constant (depending on d). This will allow us to consider quantities depending polynomially on r_i as very small compared to r_{i+1}. We will also have auxiliary degree parameters $D_1, D_2, \ldots, D_d$, where

$$D_i = O\left(r_i^{1/(d-i+1)}\right).$$

At the beginning of the ith step, $i = 1, 2, \ldots, d$, we will have the following objects:

- A complex variety V_{i-1}, which may be reducible, but such that all irreducible components have dimension $d - i + 1$. Initially, for $i = 1$, $V_0 = \mathbb{C}^d$.
- A set $Q_{i-1} \subseteq P \cap V_{i-1}$, the current "exceptional set" that still needs to be partitioned. For $i = 1$, $Q_0 = P$.

We also have

$$\deg V_{i-1} \le \Delta_{i-1} := D_1 D_2 \cdots D_{i-1}.$$

In the ith step, we apply the key lemma to V_{i-1} and Q_{i-1} with $r = r_i$ (and $k = d - i + 1$). This yields a real $(1/r_i)$-partitioning polynomial g_i for Q_{i-1} of degree at most $D_i = O\left(r_i^{1/(d-i+1)}\right)$ that does not vanish identically on any of the irreducible components of V_{i-1}. (For $i = 1$, this is just an application of the original Guth–Katz polynomial partition theorem.)

Let $S_{i1}, \ldots, S_{it_i}$ be the connected components of $(V_{i-1} \cap \mathbb{R}^d) \setminus Z(g_i)$, and let $P_{ij} := S_{ij} \cap Q_{i-1}$ (these are the sets as in Theorem 1.3). For every j we have $|P_{ij}| \le |Q_{i-1}|/r_i \le n/r_i$ since g_i is a $(1/r_i)$-partitioning polynomial. We also have the new exceptional set $Q_i := Q_{i-1} \cap Z(g_i)$.

Finally, we set $V_i := V_{i-1} \cap Z_{\mathbb{C}}(g_i)$. Since g_i does not vanish identically on any of the irreducible components of V_{i-1}, all irreducible components of V_i are $(d-i)$-dimensional by Lemma 1.8, and the sum of their degrees, which equals $\deg V_i$, is at most

$$\deg(V_{i-1}) \deg(g_i) \le \Delta_{i-1} D_i \le D_1 D_2 \cdots D_i = \Delta_i$$

as needed for the next inductive step. This finishes the ith partitioning step.

After the dth step, we end up with a 0-dimensional variety V_d, whose irreducible components are points, and their number is $\deg V_d \le \Delta_d$, a quantity polynomially bounded in r. The set Q_d is the exceptional set P^* in Theorem 1.3, and $|Q_d| \le |V_d| = \deg V_d \le \Delta_d$.

The crossing number. It remains to prove the bounds on the number of the sets S_{ij} crossed by X as in parts (i) and (ii) of the theorem.

First let $X = Z(h)$ be a hypersurface of degree $D_0 = O(1)$ as in (i). For $i = 1$, we actually get that X *intersects* at most $O\left(r_1^{1-1/d}\right)$ of the S_{1j}, because the number of the

S_{1j} intersected by X is no larger than the number of connected components of $X \setminus Z(g_1)$. By the Barone–Basu theorem (Theorem 1.20), the number of these components is bounded by $O((\deg h)(\deg g_1)^{d-1}) = O\left(D_0 D_1^{d-1}\right) = O\left(r_1^{1-1/d}\right)$ as claimed.

Now let $i \geq 2$. We want to bound the number of the sets S_{ij} crossed by X. Let $U_1, \ldots, U_b$ be the irreducible components of V_{i-1} whose real points are not completely contained in X; that is, satisfying $U_\ell \cap \mathbb{R}^d \not\subseteq X$. We have $b \leq \deg V_{i-1} \leq \Delta_{i-1}$.

For every j such that X crosses S_{ij}, let us fix a point $y_j \in S_{ij} \setminus X$ and another point $z_j \in S_{ij} \cap X$ (they exist by the definition of crossing). Since S_{ij} is path-connected, there is also a path $\gamma_j \subseteq S_{ij}$ connecting y_j to z_j.

Let z_j^* be the first point of X on γ_j when we go from y_j towards z_j. We observe that z_j^* lies in some U_ℓ. Indeed, points on γ_j just before z_j^* lie in V_{i-1} (since $S_{ij} \subseteq V_{i-1}$) but not in X, hence they lie in some U_ℓ, and U_ℓ, being an algebraic variety, is closed in the Euclidean topology.

For any given U_ℓ, a connected component of $(U_\ell \cap \mathbb{R}^d \cap X) \setminus Z(g_i)$ may contain at most one of the z_j^* (since the S_{ij} are separated by $Z(g_i)$). Therefore, the number of the S_{ij} crossed by X is no more than

$$\sum_{\ell=1}^{b} \#(W_\ell \setminus Z(g_i)),$$

where $W_\ell := U_\ell \cap \mathbb{R}^d \cap X$, and $\#$ denotes the number of connected components.

Since U_ℓ is irreducible and X does not contain all of its real points, the polynomial h defining X does not vanish on U_ℓ, and thus $U_\ell \cap Z_{\mathbb{C}}(h)$ is a proper subvariety of U_ℓ of (complex) dimension $\dim U_\ell - 1 = d - i$. Hence, by Lemma 1.7, the real variety $W_\ell = (U_\ell \cap Z_{\mathbb{C}}(h)) \cap \mathbb{R}^d$ also has (real) dimension at most $d - i$.

By Theorem 1.9, we have $U_\ell = Z_{\mathbb{C}}(f_1, \ldots, f_m)$ for some, generally complex, polynomials of degree at most $\deg U_\ell \leq \Delta_{i-1}$. Thus W_ℓ is the real zero set of the real polynomials h, $f_1\overline{f}_1, \ldots, f_m\overline{f}_m$. These polynomials have degrees bounded by $\max(D_0, 2\Delta_{i-1}) = O(\Delta_{i-1})$.

By the Barone–Basu theorem again, the number of components of $W_\ell \setminus Z(g_i)$ is at most

$$O(\Delta_{i-1}^{d-\dim W_\ell} D_i^{\dim W_\ell}) = O(\Delta_{i-1}^{d} D_i^{d-i}) = O\left(\Delta_{i-1}^{d} r_i^{1-1/(d-i+1)}\right).$$

The total number of the S_{ij} crossed by X is then bounded by Δ_{i-1} times the last quantity, i.e., by $O\left(\Delta_{i-1}^{d+1} r_i^{1-1/(d-i+1)}\right) = O\left((D_1 D_2 \cdots D_{i-1})^{d+1} r_i^{1-1/(d-i+1)}\right)$. Since $r_i = r_{i-1}^c$, we can make $(D_1 D_2 \cdots D_{i-1})^{d+1}$ smaller than any fixed power of r_i, and hence we can bound the last estimate by $O\left(r_i^{1-1/d}\right)$ (recall that $i \geq 2$), which finishes the proof of part (i) of the theorem.

For part (ii), the argument requires only minor modifications. Now X is a variety of dimension $k \leq d-2$ defined by real polynomials of degree at most $D_0 = O(1)$.

We have $\dim V_{i-1} = d - i + 1$, and for $\dim X = k \leq d - i$ we simply count the components of $X \setminus Z(g_i)$, as we did for part (i) in the case $i = 1$. This time we obtain the bound $O\left(D_0^{d-\dim X} D_i^{\dim X}\right) = O\left(D_i^k\right) = O\left(r_i^{k/(d-i+1)}\right)$.

The exponent $\frac{k}{d-i+1}$ increases with i, and thus it is the largest for $d - i = k$, in which case the bound is $O\left(r_i^{1-1/(k+1)}\right)$. (This is the critical case; for all of the other i we get a better bound.)

For $k \geq d-i+1$, we argue as in part (i): letting $U_1, \ldots, U_b$ be the irreducible components of V_{i-1} with $U_\ell \cap \mathbb{R}^d \not\subseteq X$ and $W_\ell := U_\ell \cap \mathbb{R}^d \cap X$, the number of the S_{ij} crossed by X is bounded by $\sum_{\ell=1}^{b} \#(W_\ell \setminus Z(g_i))$, and each W_ℓ has (real) dimension at most $\dim V_{i-1} - 1 = d - i$. The number of components of $W_\ell \setminus Z(g_i)$ is again bounded, by the Barone–Basu theorem, by $O\left(\Delta_{i-1}^{d} r_i^{1-1/(d-i+1)}\right)$, and the sum over all W_ℓ is $O\left(\Delta_{i-1}^{d+1} r_i^{1-1/(d-i+1)}\right)$. For every fixed $\delta > 0$, we can choose the constant c in the inductive definition of the r_i so large that $\Delta_{i-1}^{d+1} \leq r_i^{\delta}$, and so the previous bound is no more than $O\left(r_i^{1-1/(d-i+1)+\delta}\right)$.

The exponent $1 - \frac{1}{d-i+1}$ is maximum for $d-i+1 = k$, in which case our bound is $O\left(r_i^{1-1/k+\delta}\right)$. By letting $\delta := \frac{1}{k} - \frac{1}{k+1}$, we bound this by $O\left(r_i^{1-1/(k+1)}\right)$. This concludes the proof of Theorem 1.3.

The proof of Theorem 1.4 goes along the same lines, the only difference is that in the dth step, when constructing V_d from V_{d-1}, we use Lemma 1.18 instead of the key lemma. The assumption $r^K \leq n$ implies that $r_d \leq n$ and hence the assumption of Lemma 1.18 is satisfied. Clearly, $P^* = Q_d = Q_{d-1} \cap Z(g_d) = \emptyset$, where g_d is a real $(1/r_d)$-partitioning polynomial given by Theorem 1.18.

1.7 Algorithmic aspects of Theorem 1.3

The goal of this section is to prove Theorem 1.6. In order to make the proof of Theorem 1.3 algorithmic, we need to compute both with real and complex varieties. A variety V, both in the real and complex cases, is represented by a finite list $f_1, \ldots, f_m$ of polynomials such that $V = Z(f_1, \ldots, f_m)$.

The size of such a representation is measured as $m + \sum_{i=1}^{m} \deg f_i$. It would perhaps be more adequate to use $\binom{\deg f_i + d}{d}$, the number of monomials in a general d-variate polynomial of degree $\deg f_i$, instead of just $\deg f_i$, but since we consider d constant, both quantities are polynomially equivalent.

If we want to pass from a complex V defined by generally complex polynomials $f_1, \ldots, f_m$ to the real variety $V \cap \mathbb{R}^d$, we use the trick already mentioned: $V \cap \mathbb{R}^d$ is defined by the real polynomials $f_1\overline{f}_1, \ldots, f_m\overline{f}_m$.

To make the construction in Theorem 1.3 algorithmic, besides some obvious steps (such as testing the membership of a point in a variety, which is done by substituting the point coordinates into the defining polynomials), we need to implement the following operations:

(A) Given a variety V in $\mathbb{C}^d$ of dimension k, $1 \leq k \leq d-1$, such that all irreducible components of V have dimension k, and a finite point set $Q \subset V \cap \mathbb{R}^d$, compute a real projection $\pi \colon \mathbb{C}^d \to \mathbb{C}^k$ as in Corollary 1.17, i.e., such that $\pi(V_j) = \mathbb{C}^k$ for all irreducible components V_j of V, and $|\pi(Q)| = |Q|$.

(B) Given a finite point set $Q \subset \mathbb{R}^k$, $k \leq d$, construct a $\frac{1}{r}$-partitioning polynomial of degree $O\left(r^{1/k}\right)$ (as in the proof of the key lemma).

(C) Given a complex variety V and a polynomial g, compute $V \cap Z_{\mathbb{C}}(g)$.

For (A), we follow the proof of Corollary 1.17; i.e., we compute π as the composition $\pi'_{k+1}\circ\cdots\circ\pi'_d$, where $\pi'_i\colon \mathbb{C}^i \to \mathbb{C}^{i-1}$ sends $(x_1,\ldots,x_i)$ to $(x_1+\lambda_{i,1}x_i,\ldots,x_{i-1}+\lambda_{i,i-1}x_i)$, with the λ_{ij} chosen independently at random from the uniform distribution on $[0,1]$, say (or, if we do not want to assume the capability of generating such random reals, we can still choose them as random integers in a sufficiently large range). The composed π will work almost surely (or, if we use large random integers, with high probability—this can be checked using the Schwartz–Zippel lemma).

In order to verify that a particular π works, we verify the condition in the projection theorem (Theorem 1.15) and the condition on the size of $\pi'_i(Q)$ for each π'_i separately. To this end, we compute the projected varieties $V_i := \pi'_{i+1}\circ\cdots\circ\pi'_d(V)$ in $\mathbb{C}^i$; initially $V_d = V$.

The projections can be computed in a standard way using Gröbner bases w.r.t. the lexicographic ordering; see [CLO07]. Namely, we suppose that V_i has already been computed. We make the substitution $x'_j := x_j + \lambda_{ij}x_i$, where the λ_{ij} are those used in $\pi^{(i)}$ and $\lambda_{ii} = 0$; this transforms the list of polynomials defining V_i into another list of polynomials in the new variables $x'_1,\ldots,x'_i$. Since $1 \le \dim V_i \le d-1$, it follows that all the polynomials in the list have degree at least one. Thus, by Theorem 1.19, we compute a Gröbner basis G_i of the ideal generated by these new polynomials, with respect to the lexicographic ordering, where the ordering puts the variable x_i first.

If G_i contains no polynomial whose leading term is a power of x_i (as in the projection theorem), or if $|\pi_i(Q)| \ne |Q|$, then we discard π_i, generate a new one, and repeat the test. If G_i does contain such a polynomial, then we take all polynomials in G_i that do not contain x_i, and these define the variety $V_{i-1} = \pi'_i(V_i)$ in $\mathbb{C}^{i-1}$. Indeed, recall that by [CLO07, Thm. 3.1.2], if G is a Gröbner basis of $I \subseteq \mathbb{C}[x_1,\ldots,x_d]$ then $G \cap \mathbb{C}[x_1,\ldots,x_{d-1}]$ is a Gröbner basis of $I \cap \mathbb{C}[x_1,\ldots,x_{d-1}]$. The claim now follows from the projection theorem.

Thus, the computation of π takes a constant number of Gröbner basis computations and the expected number of repetitions is a constant. (In practice, the coordinate projection forgetting the last $d-k$ coordinates will probably work most of the time; then only one Gröbner basis computation is needed to verify that it works.)

For operation (B), constructing a partitioning polynomial for points in $\mathbb{R}^k$, we use a (randomized) algorithm from [AMS13, Thm. 1.1], which runs in expected time $O\left(|Q|r + r^3\right)$ for fixed k. Since each point of the original input set P participates in no more than d of these operations, and the value of r in each of these cases is bounded by a polynomial function of the original parameter r in the theorem, the total time spent in all of the operations (B) in the construction is bounded by $O(nr^C)$ for a constant C.

Operation (C), intersecting a complex variety with $Z(g)$, is trivial in our representation, since we just add g to the list of the defining polynomials of V.

This finishes the implementation of the operations, and now we need to substantiate the claims about the number and form of the sets S_{ij}. We recall that each S_{ij} is obtained as a cell in the arrangement of $Z(g_i)$ within V_{i-1}. The degrees of g_i and of the polynomials defining V_{i-1} are bounded by a polynomial in r. Then by Theorem 1.21, we get that each S_{ij} is defined by at most r^C polynomials of degree at most r^C, and is computed in r^C time. The number of the S_{ij} is polynomially bounded in r as well.

Finally, we need to consider a range $\gamma \in \Gamma_{d,D_0,s}$. By definition, γ is a Boolean combination of $\gamma_1,\ldots,\gamma_s$, where $\gamma_\ell = \{x \in \mathbb{R}^d : h_\ell(x) \ge 0\}$, with a polynomial h_ℓ of degree at most D_0, and moreover, if γ crosses a path-connected set A, then at least one of the varieties $X_\ell = Z(h_\ell)$ crosses A. It follows that the crossing number for γ

is no more than s-times the bound in Theorem 1.3(i). This concludes the proof of Theorem 1.6.

1.8 The range searching result

The derivation of the range searching result, Theorem 1.5, from Theorem 1.6, is by a standard construction of a partition tree as in [Mat92, AMS13], and here we give it for completeness (and also to illustrate its simplicity).

Proof of Theorem 1.5. Given $d, D_0, s, \varepsilon > 0$ and a set $P \subset \mathbb{R}^d$, we choose a sufficiently large $n_0 = n_0(d, D_0, s, \varepsilon)$ and a sufficiently small parameter $\eta = \eta(d, D_0, s, \varepsilon) > 0$, and we construct a partition tree $\mathcal{T}$ for P recursively as follows:

If $|P| \leq n_0$, $\mathcal{T}$ consists of a single node storing a list of the points of P and their weights.

For $|P| > n_0$, we choose $r := n^\eta$ and we construct P^*, the P_{ij}, and the S_{ij} as in Theorem 1.3. The root of $\mathcal{T}$ stores (the formulas defining) the S_{ij}, the total weight of each P_{ij}, and the points of P^* together with their weight. For each i and j, we make a subtree of the root node, which is a partition tree for P_{ij} constructed recursively by the same method.

By Theorem 1.6, the construction of the root node of $\mathcal{T}$ takes expected time $O\left(nr^C\right) = O\left(n^{1+C\eta}\right)$. The total preprocessing time $T(n)$ for an n-point P obeys the recursion, for $n > n_0$, $T(n) \leq O\left(n^{1+C\eta}\right) + \sum_{i,j} T(n_{ij})$, with $\sum_{i,j} n_{ij} \leq n$ and $n_{ij} \leq n/r = n^{1-\eta}$, whose solution is $T(n) \leq O\left(n^{1+C\eta}\right)$. A similar simple analysis shows that the total storage requirement is $O(n)$.

Let us consider answering a query with a query range $\gamma \in \Gamma_{d,D_0,s}$. We start at the root of $\mathcal{T}$ and maintain a global counter which is initially set to 0. We test the points of the exceptional set P^* for membership in γ one by one and increment the counter accordingly in $r^{O(1)}$ time. Then, for each i, j, we distinguish three possibilities:

(i) If $S_{ij} \cap \gamma = \emptyset$, we do nothing.

(ii) If $S_{ij} \subseteq \gamma$, we add the total weight of the points of P_{ij} to the global counter.

(iii) Otherwise, we recurse in the subtree corresponding to P_{ij}, which increments the counter by the total weight of the points of $P_{ij} \cap \gamma$.

The three possibilities above can be distinguished, for given S_{ij}, by constructing the arrangement of the zero sets of the polynomials defining S_{ij} plus the polynomials defining γ, according to Theorem 1.21. The total time, for all i, j together, is $r^{O(1)}$.

Since, by Theorem 1.6, γ together crosses at most $O\left(r_i^{1-1/d}\right)$ of the S_{ij}, possibility (iii) occurs, for given i, for at most $O\left(r_i^{1-1/d}\right)$ values of j. We thus obtain the following recursion for the query time $Q(n)$, with the initial condition $Q(n) = O(1)$ for $n \leq n_0$:

$$Q(n) \leq n^{C'\eta} + \sum_{i=1}^{d} O\left(r_i^{1-1/d}\right) Q\left(n/r_i\right), \qquad n^\eta \leq r_i \leq n^{K\eta},$$

where C' and K are constants independent of η. A simple induction on n verifies that this implies, for $\eta \leq (1 - 1/d)/C'$, $Q(n) = O\left(n^{1-1/d} \log^B n\right)$ as claimed. □

1.9 Remark: On (not) computing irreducible components

For the algorithmic part, it is important that we do not need to compute the irreducible components of the varieties V_i (although we use the irreducible components in the proof of our multilevel partition theorem).

There are several algorithms in the literature for computing irreducible components of a given complex variety (e.g., [EM99]). However, these algorithms need factorization of multivariate polynomials over $\mathbb{C}$ as a subroutine (after all, factoring a polynomial corresponds to computing irreducible components of a hypersurface).

Polynomial factorization is a well-studied topic, with many impressive results; see, e.g., [Kal92] for a survey. In particular, there are algorithms that work in polynomial time, assuming the dimension fixed, but only in the Turing machine model. Adapting these algorithms to the Real RAM model, which is common in computational geometry and which we use, encounters some nontrivial obstacles—we are grateful to Erich Kaltofen for explaining this issue to us.

It may perhaps be possible to overcome these obstacles by techniques used in real algebraic geometry for computing in abstract real-closed fields (see [BPR03]), but this would need to be worked out carefully. Then one could probably obtain rigorous complexity bounds on computing irreducible components of a complex variety, hopefully polynomial in fixed dimension; we find this question of independent interest.

1.10 A version of the key lemma for an irreducible variety

Here we prove Lemma 1.14. For the convenience of the reader, we recall the lemma:

Lemma 1.14. *Let $V \subseteq \mathbb{C}^d$ be an irreducible complex algebraic variety of dimension $k \geq 1$ and degree Δ. Let $Q \subset V \cap \mathbb{R}^d$ be a finite point set, and let $r > 1$ be a parameter. Then there exists a real $\frac{1}{r}$-partitioning polynomial g for Q of degree at most $O\left(\Delta + \left(\frac{r}{\Delta}\right)^{1/k}\right)$ that does not vanish identically on V.*

Note that for $k = d$ the lemma reduces to Theorem 1.1, and for $k = d - 1$ to Theorem 1.2. Hence we may assume that $k \leq d - 2$.

Before proving the lemma, we first sketch the idea. The very rough idea is to find a linear map sending V to an irreducible hypersurface having same degree as V (Lemma 1.23). Since the hypersurface is defined by a single polynomial, we can apply Theorem 1.2 to the image of the given set Q and get a $(1/r)$-partitioning polynomial. Finally, we pull it back to obtain $(1/r)$-partitioning polynomial for Q.

We note that in order to overcome technical difficulties, we will mainly work with projective varieties. Now we present the approach in more detail.

1.10.1 Projections and degree of the variety

Let $H \subset \mathrm{P}\mathbb{C}^d$ be a hyperplane and let $p \in \mathrm{P}\mathbb{C}^d$ be a point not lying on H. We consider the map $\pi_p \colon \mathrm{P}\mathbb{C}^d \setminus \{p\} \to H \cong \mathrm{P}\mathbb{C}^{d-1}$ given by $q \mapsto \overleftrightarrow{qp} \cap H$, that is, sending point $q \in \mathrm{P}\mathbb{C}^d, q \neq p$ to the point of intersection of the unique line $\overleftrightarrow{pq}$ with the hyperplane H. Let $X \subset \mathrm{P}\mathbb{C}^d$ be a variety not containing the point p. If we restrict the map π_p to X, we get a regular map[9] whose image $\pi_p(X)$ is called *projection from the point p to the hyperplane H*. Moreover, $\pi_p(X)$ is again a variety [Har92, Thm 3.5].

The degree of a k-dimensional irreducible variety $X \subset \mathrm{P}\mathbb{C}^d$ can be also defined via projections. Let us assume that X is nondegenerate, that is, it does not lie in any hyperplane. If $k = d-1$, then X is a zero set of a single irreducible polynomial f and, as we have already seen in Section 1.2, $\deg X = \deg f$ (see, e.g., [Har77, Exercise I.2.8]).

Let $k \leq d-2$ and let us assume that X and $\pi_{p_1}(X)$ are birational[10], where $p_1 \notin X$ and π_{p_1} is a projection from p_1 to some hyperplane $H_1 \subset \mathrm{P}\mathbb{C}^d$, $p_1 \notin H_1$. Let $p_2 \in H_1 \setminus \pi_p(X)$ be such a point, that $\pi_{p_1}(X)$ and $\pi_{p_2}(\pi_{p_1}(X))$ are birational, where π_{p_2} stands for a projection from p_2 to some $H_2 \cong \mathrm{P}\mathbb{C}^{d-2}$. Continuing this process, that is, projecting successively from suitable points, we get a map $\pi := \pi_{p_{d-k-1}} \circ \cdots \circ \pi_{p_1}$ such that X is birational with $\pi(X)$, which is a hypersurface in $\mathrm{P}\mathbb{C}^{k+1}$; and we may define the degree of X as the degree of this hypersurface. By Harris [Har92, Chapter 18] this is a correct definition and it is equivalent to the one via Hilbert polynomials and also to the one given in Section 1.2. It remains to decode, what exactly are the suitable points $p_1, p_2, \ldots, p_{d-k-1}$. Segre [Seg36] provided the answer: He showed that "suitable" means "generic". More precisely, he showed that if $X \subset \mathrm{P}\mathbb{C}^d$ is an irreducible, nondegenerate algebraic variety of dimension $k < d-1$, then there is a union of finitely many linear subspaces $\mathfrak{S}(X)$ of $\mathrm{P}\mathbb{C}^d$ such that all of its irreducible components have dimension strictly less than k and for any $p \notin \mathfrak{S}(X) \cup X$, the projection $\pi_p \colon X \to \mathrm{P}\mathbb{C}^{d-1}$ is generically one-to-one, that is, X and $\pi_p(X)$ are birational. (See also [CC01].) The variety $\mathfrak{S}(X)$ is called *Segre locus* of X.

We summarize all the preceding considerations into the following lemma:

Lemma 1.22. *Let $X \subseteq \mathrm{P}\mathbb{C}^d$ be a nondegenerate irreducible variety of dimension $k \geq 1$ and let $p_1 \notin X \cup \mathfrak{S}(X)$ be any point. Then X and $\pi_{p_1}(X)$ are birational, where $\pi_{p_1} \colon \mathrm{P}\mathbb{C}^d \setminus \{p\} \to \mathrm{P}\mathbb{C}^{d-1}$ is a projection from p_1. Let $p_i \notin \pi_i(X) \cup \mathfrak{S}(\pi_i(X))$, where $\pi_i := \pi_{p_i} \circ \cdots \circ \pi_{p_1}$. Then X is birational to an irreducible hypersurface $\pi_{d-k-1}(X) \subset \mathrm{P}\mathbb{C}^{k+1}$ of degree* $\deg X$.

The last ingredient needed for a proof of Lemma 1.14 is the following lemma:

Lemma 1.23. *Let $V \subseteq \mathbb{C}^d$ be an irreducible complex variety of dimension $1 \leq k \leq d-2$. Then there is a linear map $\pi \colon \mathbb{C}^d \to \mathbb{C}^{k+1}$ such that $\pi(\mathbb{R}^d) \subseteq \mathbb{R}^{k+1}$, $\pi(V)$ is an irreducible hypersurface and* $\deg \pi(V) = \deg V$.

Proof. Embed $\mathbb{C}^d$ into $\mathrm{P}\mathbb{C}^d$ via $(x_1, \ldots, x_d) \mapsto [1 \colon x_1 \colon \ldots \colon x_d]$ and consider the projective closure $\overline{V} \subset \mathrm{P}\mathbb{C}^d$ of V. We recall that $\overline{V}$ is irreducible, $\dim \overline{V} = \dim V$ and

[9]Let $X \subseteq \mathrm{P}\mathbb{C}^d$ be a variety. In order to properly define a regular map, some preliminary definitions are needed: we say that a function $f : X \to \mathbb{C}$ is *regular at a point* $p \in X$ if there is an open neighborhood U with $p \in U \subseteq X$, and homogeneous polynomials $g, h \in \mathbb{C}[x_0, \ldots, x_d]$, of the same degree, such that h is nowhere zero on U and $f = g/h$ on U. We say that f is *regular on* X if it is regular at every point. If $X, Y \subseteq \mathrm{P}\mathbb{C}^d$ are two varieties, a *regular map* $\varphi \colon X \to Y$ is a continuous map such that for every open set $V \subseteq Y$, and for every regular function $f \colon V \to \mathbb{C}$, the function $f \circ \varphi \colon \varphi^{-1}(V) \to \mathbb{C}$ is regular. We note that we consider the Zariski topology, that is, topology in which the open sets are the complements of varieties.

[10] We say that two irreducible varieties X, Y are *birational* if there exists a birational map between them. A *birational map* $\varphi \colon X \to Y$ is a rational map which admits an inverse, namely a rational map $\psi \colon Y \to X$ such that $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$. A *rational map* $\varphi \colon X \to Y$ is defined to be an equivalence class of pairs (U, φ_U) with U a dense open subset of X and $\varphi_U \colon U \to Y$ a regular map, where two such pairs (U, φ_U) and (V, φ_V) are equivalent if φ_U and φ_V agree on $U \cap V$.

$\deg V = \deg \overline{V}$. We can also assume that $\overline{V}$ is nondegenerate, otherwise we just appropriately decrease the dimension d.

First observe, that we can pick a point $p_1 \in \mathrm{P}\mathbb{C}^d$ such that both conditions are satisfied:

(i) $p_1 \notin \overline{V} \cup \mathfrak{S}(\overline{V})$,

(ii) $p_1 = [0\colon a_1\colon \cdots : a_d]$, where $a_1, \ldots, a_d \in \mathbb{R}$.

Indeed, $\overline{V} \cup \mathfrak{S}(\overline{V})$ is a variety of dimension at most $d-2$ since $\dim \overline{V} \leq d-2$, $\dim \mathfrak{S}(\overline{V}) < d-2$, and the dimension of a variety is equal to the maximal dimension of its irreducible components; the rest is clear.

Now we consider a suitable linear change of coordinates. We note that, similarly as in the affine case, the change of coordinates can be represented by a regular matrix. Consider a linear change of coordinates τ_1 represented by a (regular) matrix $A = (a_{ij})_{i,j=1}^{d+1}$ such that, moreover, the following three conditions are satisfied:

(a) $\tau_1(p_1) = [0\colon 0\colon \cdots : 1]$,

(b) $a_{11} = 1$, $a_{1j} = a_{j1} = 0$ for $j = 2, \ldots, d+1$,

(c) $a_{ij} \in \mathbb{R}$ for $2 \leq i, j \leq d+1$.

Since p_1 has all coordinates real, it follows that such change of coordinates always exists, however, it is not determined uniquely. We say that a point $p \in \mathrm{P}\mathbb{C}^d$ is *affine* if its first coordinate is nonzero. Note that the geometric meaning of condition (b) is that $\tau_1(p)$ is affine if and only if p is affine. Let us fix one particular coordinate change satisfying (a)-(c). The projection $\pi_1' := \pi_{\tau_1(p_1)}$ is then given by $[x_0\colon \cdots : x_d] \mapsto [x_0\colon \ldots : x_{d-1}]$. Set $\pi_1 := \pi_1' \circ \tau_1$. Using Lemma 1.22, it follows that $\overline{V}$ and $\pi_1(\overline{V})$ are birational. Hence $\pi_1(\overline{V})$ is irreducible, nondegenerate and $\dim \pi_1(\overline{V}) = \dim \overline{V}$. We repeat the process and pick a point $p_2 \notin \pi_1(\overline{V}) \cup \mathfrak{S}(\pi_1(\overline{V}))$ having first coordinate zero and all the others real. Again, we consider a linear change of coordinates τ_2 satisfying (a)-(c), however, now with p_2 instead of p_1 and $i, j \leq d$. We set $\pi_2 := \pi_{\tau_2(p_2)} \circ \tau_2$. Clearly, $\pi_1(\overline{V})$ and $\pi_2(\pi_1(\overline{V}))$ are birational. It follows that we can repeat the process and at the end, we set $\pi' = \pi_{d-k-1} \circ \cdots \circ \pi_1$.

Note that the map π' can be viewed as a composition of corresponding coordinate changes with a projection given by $[x_0\colon \cdots : x_d] \mapsto [x_0\colon \cdots : x_{k+1}]$. Since the coordinate change does not change a degree of a variety, it follows, by Lemma 1.22, that $\pi'(\overline{V})$ is an irreducible hypersurface of degree $\deg \overline{V}$. Moreover, by condition (b), π' maps affine points to the affine ones. The map π' induces a map $\pi\colon \mathbb{C}^d \to \mathbb{C}^{k+1}$, which is just a restriction of π' to the affine part (points with first coordinate nonzero). It follows that $\pi(V) = \pi'(\overline{V}) \cap \mathbb{C}^d$, hence $\pi(V)$ is a variety. It also follows that $\dim \pi(V) = \dim \pi'(\overline{V}) = k$. Moreover, since $\pi'(\overline{V}) \supseteq \overline{\pi(V)}$, $\dim \overline{\pi(V)} = \dim \pi(V) = \dim \pi'(\overline{V})$ and $\pi'(\overline{V})$ is irreducible, it follows that $\overline{\pi(V)}$ is irreducible as well and so is $\pi(V)$. Furthermore, $\deg \pi(V) = \deg \pi'(\overline{V}) = \deg \overline{V} = \deg V$.

Finally, from the conditions (b) and (c) in the definition of π' easily follows that $\pi(\mathbb{R}^d) \subseteq \mathbb{R}^{k+1}$. $\square$

Now we are ready to prove Lemma 1.14.

Proof of Lemma 1.14. Recall that we may assume that $k \leq d-2$. Given the k-dimensional irreducible complex variety V and the point set $Q \subset V \cap \mathbb{R}^d$ as in Lemma 1.14, we consider a projection $\pi\colon \mathbb{C}^d \to \mathbb{C}^{k+1}$ as in Lemma 1.23. Since $\pi(\mathbb{R}^d) \subseteq \mathbb{R}^{k+1}$, we can regard $Q' := \pi(Q)$ as a subset of $\mathbb{R}^{k+1}$. More precisely, Q' is a *multiset* in general, since π may send several points to the same point. (We can avoid such coincidences in the choice of π in a similar manner as we did in the proof of Corollary 1.17.)

By Lemma 1.23, $\pi(V)$ is an irreducible hypersurface H in $\mathbb{C}^{k+1}$ of degree Δ; we denote this hypersurface by H. In other words, $H = Z_{\mathbb{C}}(f)$, where $f \in \mathbb{C}[x_1, \ldots, x_{k+1}]$ is an irreducible polynomial of degree Δ. Our aim now is to find an irreducible polynomial $f' \in \mathbb{R}[x_1, \ldots, x_{k+1}]$, such that $Z_{\mathbb{C}}(f) \cap \mathbb{R}^{k+1} = Z(f')$. Recall that if h is real, then $Z_{\mathbb{C}}(h) \cap \mathbb{R}^{k+1} = Z(h)$. If f is a complex scalar multiple of a real polynomial, that is, $f = ag$, where $a \in \mathbb{C}, g \in \mathbb{R}[x_1, \ldots, x_{k+1}]$, we set $f' := g$, otherwise $f' := f\overline{f}$. Since $Z(f\overline{f}) = H \cap \mathbb{R}^{k+1} = Z_{\mathbb{C}}(f) \cap \mathbb{R}^{k+1}$, it remains to show that f' is irreducible over $\mathbb{R}$. For $f' = g$ is the irreducibility clear (g is irreducible over $\mathbb{C}$ and hence also over $\mathbb{R}$). Thus let us assume for contradiction that $f' = f\overline{f}$ is reducible over $\mathbb{R}$. There exist real polynomials r, s such that $f' = rs$. Since f is irreducible and it is not a scalar multiple of a real polynomial, it follows that rs divide $\overline{f}$. This is a contradiction, since complex conjugation is an automorphism on $\mathbb{C}$ and hence $\overline{f}$ is irreducible as well. We conclude that f' is a real irreducible polynomial of degree D, where $\Delta \leq D \leq 2\Delta$.

Let $r > 1$ be the parameter from the statement. We apply Theorem 1.2 on the polynomial f' of degree $\Theta(\Delta)$, the set Q' and r. We get a $\frac{1}{r}$-partitioning polynomial g' coprime with f' of degree $O(\Delta + (\frac{r}{\Delta})^{1/k})$. We note that Theorem 1.2 works for multisets without any change (because the ham-sandwich theorem used in the proof can be easily adapted to multisets, see the proof of Theorem 3.1.2 in [Mat03]).

We define a polynomial $g \in \mathbb{R}[x_1, \ldots, x_d]$ as the pullback of g', i.e., $g(x) := g'(\pi(x))$. We have $\deg g = \deg g'$ since π is linear and surjective.

Moreover, g is a $\frac{1}{r}$-partitioning polynomial for Q, since if $\pi(q)$ and $\pi(q')$ lie in different components of $\mathbb{R}^{k+1} \setminus Z(g')$, then q and q' lie in different components of $\mathbb{R}^d \setminus Z(g)$ (indeed, if not, a path γ connecting q to q' and avoiding $Z(g)$ would project to a path γ' connecting $\pi(q)$ to $\pi(q')$ and avoiding $Z(g')$).

Finally, g' is coprime with f', which in other words means that g' does not vanish identically on $H \cap \mathbb{R}^{k+1}$, and the polynomial g does not vanish identically on V. The lemma follows. □

Chapter 2

Lower bounds on geometric Ramsey functions

2.1 Introduction

Ramsey's theorem and the classical Ramsey function. A classical and fundamental theorem of Ramsey claims that for every n there is a number N such that for every 2-coloring of the edge set of the complete graph K_N on N vertices there is a *homogeneous* subset of n vertices, meaning that all edges in the complete subgraph induced by these n vertices have the same color. More generally, for every k and n there exists N such that if the set of all k-tuples of elements of an N-element set X is colored by two colors, then there exists an n-element homogeneous $Y \subseteq X$, with all k-tuples from Y having the same color. Let $R_k(n)$ stand for the smallest N with this property.

Considering k fixed and n large, the best known lower and upper bounds for the Ramsey function $R_k(n)$ are of the form $R_2(n) = 2^{\Theta(n)}$ and, for $k \geq 3$,

$$\mathrm{twr}_{k-1}(\Omega(n^2)) \leq R_k(n) \leq \mathrm{twr}_k(O(n)),$$

where the tower function $\mathrm{twr}_k(x)$ is defined by $\mathrm{twr}_1(x) = x$ and $\mathrm{twr}_{i+1}(x) = 2^{\mathrm{twr}_i(x)}$.

A widely believed, and probably very difficult, conjecture of Erdős and Hajnal [EHR65] asserts that the upper bound is essentially the truth. This is supported by known bounds for more than two colors, where the lower bound for k-tuples is also a tower of height k; see Conlon, Fox, and Sudakov [CFS13] for a recent improvement and more detailed overview of the known bounds.

Better Ramsey functions for geometric Ramsey-type results. Ramsey's theorem can be used to establish many geometric Ramsey-type results concerning configurations of points, or of other geometric objects, in $\mathbb{R}^d$. The first two examples, which up until now remain among the most significant and beautiful ones, come from a 1935 paper of Erdős and Szekeres [ES35].

The first one asserts that every sufficiently long sequence $(x_1, \ldots, x_N)$ of real numbers contains a subsequence $(x_{i_1}, x_{i_2}, \ldots, x_{i_n})$, $i_1 < i_2 < \cdots < i_n$, that is either increasing, i.e., $x_{i_1} < x_{i_2} < \cdots < x_{i_n}$, or nonincreasing, i.e., $x_{i_1} \geq x_{i_2} \geq \cdots \geq x_{i_n}$.

Ramsey's theorem for $k = 2$ yields the bound $N \leq R_2(n) \leq \mathrm{twr}_2(O(n))$ (color a pair $\{i, j\}$, $i < j$, red if $x_i < x_j$ and blue if $x_i \geq x_j$), but the result is known to hold with $N = (n-1)^2 + 1$, an exponential improvement over $R_2(n)$.

For the second of the two Erdős–Szekeres theorems mentioned above, we consider a sequence $P = (p_1, p_2, \ldots, p_N)$ of points in the plane; for simplicity, we assume that the p_i are in general position (no three collinear). If N is sufficiently large, then there is a subsequence $(p_{i_1}, \ldots, p_{i_n})$, $i_1 < i_2 < \cdots < i_n$, forming the vertex set of a convex n-gon, enumerated clockwise or counterclockwise.

This time Ramsey's theorem yields $N \leq R_3(n) \leq \mathrm{twr}_3(O(n))$, by coloring a triple $\{i, j, k\}$, $i < j < k$, red if p_i, p_j, p_k appear clockwise around the boundary of their convex hull, and blue otherwise. Again, the optimal bound is one exponential better, of order $2^{\Theta(n)}$.

It is natural to ask, what is special about the two-colorings of pairs or triples in the above two examples, what makes the Ramsey functions here considerably smaller, compared to arbitrary colorings?

One kind of a combinatorial condition for two-colorings of k-tuples implying such improved bounds was given by Fox, Pach, Sudakov, and Suk [FPSS12], and another by Eliáš and Matoušek [EM13]; both of them include the two Erdős–Szekeres results as special cases. However, a considerably more general, and probably more interesting, reason for the better Ramsey behavior of these geometric examples is that the colorings are "algebraically defined"; more precisely, they are given by *semialgebraic predicates*.

Upper bounds for semialgebraic colorings. Let $x_1, \ldots, x_k$ be points in $\mathbb{R}^d$, with $x_{i,j}$ denoting the jth coordinate of x_i; we regard the $x_{i,j}$ as variables. A *k-ary d-dimensional semialgebraic predicate* $\Phi(x_1, \ldots, x_k)$ is a Boolean combination of polynomial equations and inequalities in the $x_{i,j}$. More explicitly, there are a Boolean formula $\phi(X_1, \ldots, X_t)$ in Boolean variables $X_1, \ldots, X_t$ and polynomials $f_1, \ldots, f_t$ in the variables $x_{i,j}$, $1 \leq i \leq k$, $1 \leq j \leq d$, such that $\Phi(x_1, \ldots, x_k) = \phi(A_1, \ldots, A_t)$, where A_ℓ is true if $f_\ell(x_{1,1}, \ldots, x_{k,d}) \geq 0$ and false otherwise.

We say that a sequence $(p_1, \ldots, p_n)$ of points in $\mathbb{R}^d$ is *Φ-homogeneous* if either $\Phi(p_{i_1}, \ldots, p_{i_k})$ holds for every choice $1 \leq i_1 < \cdots < i_k \leq n$, or it does not hold for any such choice. The Ramsey function $R_\Phi(n)$ is the smallest N such that every point sequence of length N contains a Φ-homogeneous subsequence of length n.

The following general upper bound was first proved by Alon, Pach, Pinchasi, Radoičić, and Sharir [APP+05] for $k = 2$, and then generalized by Conlon, Fox, Pach, Sudakov, and Suk [CFP+14] for $k \geq 3$:

Theorem 2.1 ([APP+05, CFP+14]). *For every d, k, and a k-ary d-dimensional semialgebraic predicate Φ,*

$$R_\Phi(n) \leq \mathrm{twr}_{k-1}(n^C),$$

where C is a constant depending on d, k, Φ.[1]

Thus, the Ramsey function for k-ary semialgebraic predicates is bounded above by a tower one lower than the "combinatorial" Ramsey function $R_k(n)$. Let us note that for the case of increasing or nonincreasing subsequences ($k = 2$, $d = 1$) and subsequences in convex position ($k = 3$, $d = 2$) as above, Theorem 2.1 yields somewhat weak bounds, namely, $n^{O(1)}$ and $2^{n^{O(1)}}$ instead of n^2 and $2^{O(n)}$, respectively, but still in the right range.

[1]Actually, the constant C depends on Φ only through its *description complexity*, which Conlon et al. define as $\max(m, D)$, where m is the number of polynomials occurring in Φ and D is the maximum degree of these polynomials. Thus, the bound does not depend on the magnitude of the coefficients in the polynomials.

By very different methods, Bukh and Matoušek [BM14] obtained a doubly exponential upper bound for all one-dimensional semialgebraic predicates, for arbitrary k:

Theorem 2.2 ([BM14])**.** *For every* 1*-dimensional semialgebraic predicate* Φ *there is a constant* C *such that* $R_\Phi(n) \le \mathrm{twr}_3(Cn)$*.*

This opens an interesting possibility, namely, that the Ramsey function of d-dimensional semialgebraic predicates might be bounded by a tower whose height depends only on d (and not on k), but currently this question is wide open. But certainly it makes it interesting to study the dependence of the Ramsey function on the dimension.

Lower bounds. The classical Erdős–Szekeres result on subsequences in convex position [ES35] supplies a lower bound of $2^{\Omega(n)} = \mathrm{twr}_2(\Omega(n))$ in the setting of Theorem 2.1 for $k = 3$ and $d = 2$. Eliáš and Matoušek [EM13] constructed a reasonably natural[2] 4-ary planar semialgebraic Φ with $R_\Phi(n) \ge \mathrm{twr}_3(\Omega(n))$. This shows that for $k \le 4$, the height of the tower in Theorem 2.1 is optimal in terms of k.

For $d = 1$, [BM14] provided a one-dimensional 5-ary Φ with $R_\Phi(n) \ge \mathrm{twr}_3(\Omega(n))$, matching Theorem 2.2. Conlon et al. [CFP$^+$14] improved the arity to 4, which is optimal in view of Theorem 2.1.

Moreover, they obtained a lower bound almost matching Theorem 2.1 for an arbitrary k. Namely, for every $k \ge 4$ they constructed a d-dimensional k-ary semialgebraic predicate Φ such that $R_\Phi(n) \ge \mathrm{twr}_{k-1}(\Omega(n))$. However, the dimension d in their construction is large: $d = 2^{k-4}$.

A stronger lower bound. In this chapter we first modify (and simplify) the lower bound construction of Conlon et al. [CFP$^+$14], obtaining examples in considerably lower dimension.

Theorem 2.3. *For every* $d \ge 2$ *there is a* d*-dimensional semialgebraic predicate* Φ *of arity* $k = d + 3$ *such that*

$$R_\Phi(n) \ge \mathrm{twr}_{k-1}(\Omega(n)).$$

The proof is given in Section 2.2. In view of Theorem 2.2, the dependence of the tower height on the dimension in this result might even be optimal.

Super-order-type homogeneous subsequences. Next, we provide a natural geometric Ramsey-type theorem in $\mathbb{R}^d$ in which the Ramsey function is a tower of height d.

Let $T = (p_1, \ldots, p_{d+1})$ be an ordered $(d+1)$-tuple of points in $\mathbb{R}^d$. We recall that the *sign* (or *orientation*) of T is defined as $\mathrm{sgn}\det M$, where the jth column of the $(d+1) \times (d+1)$ matrix M is $(1, p_{j,1}, p_{j,2}, \ldots, p_{j,d})$. Geometrically, the sign is $+1$ if the d-tuple of vectors $p_1 - p_{d+1}, \ldots, p_d - p_{d+1}$ forms a positively oriented basis of $\mathbb{R}^d$, it is -1 if it forms a negatively oriented basis, and it is 0 if these vectors are linearly dependent.

We call a sequence $(p_1, p_2, \ldots, p_n)$ of points in $\mathbb{R}^d$ in general position *order-type homogeneous* if all $(d+1)$-tuples $(p_{i_1}, \ldots, p_{i_{d+1}})$, $i_1 < \cdots < i_{d+1}$, have the same

[2] By a "natural" predicate we mean here one that has a clear geometric meaning and seems reasonable to study in its own right, not only as a lower-bound example for a general result. In the case of [EM13], assuming that the considered four points $x_1, \ldots, x_4$ are numbered in the order of increasing first coordinates, the predicate asserts that x_4 lies above the graph of the unique quadratic polynomial passing through x_1, x_2, x_3.

sign (which is nonzero, by the general position assumption). Such sequences are of interest from various points of view: For example, the convex hull of an order-type homogeneous sequence is combinatorially equivalent to a cyclic polytope (see, e.g., [Zie94] for background). They can also be viewed as *discrete Chebyshev systems*; see [KS66], as well as a remark below.

By Ramsey's theorem, every sufficiently long point sequence in general position contains an order-type homogeneous subsequence of length n (we color every $(d+1)$-tuple by its sign). Letting $\mathrm{OT}_d(n)$ be the corresponding Ramsey function, we obtain $\mathrm{OT}_d(n) \le \mathrm{twr}_d(n^C)$ from Theorem 2.1. This has recently been improved to $\mathrm{OT}_d(n) \le \mathrm{twr}_d(O(n))$ by Suk [Suk14].

This upper bound is essentially tight. Until recently this was proved only for $d = 2$ (by [ES35]) and $d = 3$ [EM13]. As will be explained next, our results, together with a recent paper of Bárány, Pór, and Matoušek [BMP13], yield a matching lower bound for all d.

In this chapter we prove a lower bound for a somewhat stronger notion of homogeneity. Namely, let $\pi_j \colon \mathbb{R}^d \to \mathbb{R}^j$ denote the projection on the first j coordinates. We say that a point sequence $P = (p_1, \ldots, p_n)$ in $\mathbb{R}^d$ is *super-order-type homogeneous* if, for each $j = 1, 2, \ldots, d$, the projected sequence $\pi_j(P) = (\pi_j(p_1), \ldots \pi_j(p_n))$ is order-type homogeneous.

By iterated application of Ramsey's theorem, it can be seen that every sufficiently long point sequence in general position in $\mathbb{R}^d$ contains a super-order-type homogeneous subsequence of length n. Let $\mathrm{OT}^*_d(n)$ be the corresponding Ramsey function. We have the following lower bound, proved in Section 2.3:

Theorem 2.4. *For every* $n \ge d+1$, $\mathrm{OT}^*_d(n) \ge \mathrm{twr}_d(n-d)$.

In [BMP13] it is proved that $\mathrm{OT}^*_d(n) \le \mathrm{OT}_d(C_d n)$ for every d, where C_d is a suitable constant. Thus, we also obtain a lower bound for OT_d, which is tight up to a multiplicative constant in front of n:

Corollary 2.5. *We have* $\mathrm{OT}_d(n) \ge \mathrm{twr}_d(\Omega(n))$.

Chebyshev systems. Let A be a linearly ordered set of at least $k+1$ elements. A (real) *Chebyshev system* (also spelled Tchebycheff) on A is a system of continuous real functions $f_0, f_1, \ldots, f_k \colon A \to \mathbb{R}$ such that for every choice of elements $t_0 < t_1 < \cdots < t_k$ in A, the matrix $(f_i(t_j))_{i,j=0}^k$ has a (strictly) positive determinant. Chebyshev systems are mostly considered for A an interval in $\mathbb{R}$ with the natural ordering, the basic example being $f_i(t) = t^i$, but the case of finite A (*discrete* Chebyshev systems) has been investigated as well. The functions $f_0, \ldots, f_k$ as above form a *Markov system*, also called a *complete Chebyshev system*, if $f_0, \ldots, f_i$ is a Chebyshev system for every $i = 1, 2, \ldots, k$. Chebyshev systems are of considerable importance in several areas, such as approximation theory or the theory of finite moments; see the classical monograph of Karlin and Studden [KS66] or, e.g., Carnicer, Peña, and Zalik [CPZ98] for a more recent study.

In our setting, it is easy to check that an n-point order-type homogeneous sequence $P = (p_1, \ldots, p_n)$ in $\mathbb{R}^d$ gives rise to a Chebyshev system on $A = \{1, 2, \ldots, n\}$, by setting $f_j(i) = p_{i,j}$ for $j = 1, 2, \ldots, d$ and $f_0 \equiv 1$ (possibly with changing the sign for one of the f_i, if the signs of the $(d+1)$-tuples in P are negative), and conversely, from a discrete Chebyshev system with $f_0 \equiv 1$ we obtain an order-type homogeneous sequence. Similarly, super-order-type homogeneous sequences correspond to discrete Markov systems.

2.2 Lower bound for semialgebraic predicates in a small dimension

Here we prove Theorem 2.3. As was remarked in the introduction, our construction can be regarded as a modification of that of Conlon et al. [CFP+14, Lemma 3.1], but we give a self-contained presentation.

Stepping up. The proof proceeds by induction on d; having constructed a suitable d-dimensional k-ary semialgebraic predicate and an N-point sequence $P \subset \mathbb{R}^d$ without long Φ-homogeneous subsequences, we produce a $(d+1)$-dimensional $(k+1)$-ary semialgebraic predicate Ψ and a 2^N-point sequence $Q \subset \mathbb{R}^{d+1}$ without long Ψ-homogeneous subsequences.

Our basic tool is a classical stepping-up lemma of Erdős and Hajnal, see [GRS90] or [CFS13]. We first recall it in the standard combinatorial setting, and then we will work on transferring it to a semialgebraic setting.

Let $I = [N] := \{1, 2, \dots, N\}$, and let $\chi\colon \binom{I}{k} \to \{0,1\}$ with $k \geq 2$ be a given two-coloring of all k-tuples of I. Let $J = \{0,1\}^N$ be the set of all binary vectors of length N ordered lexicographically.

We define a coloring $\chi'\colon \binom{J}{k+1} \to \{0,1\}$ of all $(k+1)$-tuples of J. First we introduce[3] a function $\delta\colon J \times J \setminus \Delta \to I$ by

$$\delta(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \min\{i \in I : \alpha_i \neq \beta_i\}.$$

For a $(k+1)$-tuple $(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{k+1})$ of binary vectors, $\boldsymbol{\alpha}_1 <_{\text{lex}} \cdots <_{\text{lex}} \boldsymbol{\alpha}_{k+1}$, we write $\delta_\ell := \delta(\boldsymbol{\alpha}_\ell, \boldsymbol{\alpha}_{\ell+1})$. Then χ', the *stepping-up coloring* for χ, is given by

$$\chi'(\{\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{k+1}\}) := \begin{cases} \chi(\{\delta_1, \dots, \delta_k\}) & \text{if } \delta_1 < \cdots < \delta_k \text{ or } \delta_1 > \cdots > \delta_k \\ 1 & \text{if } \delta_1 < \delta_2 > \delta_3 \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Now the stepping-up lemma can be stated as follows.

Lemma 2.6 (Stepping-up lemma)**.** *If χ is a two-coloring of the k-tuples of $I := [N]$ under which I has no homogeneous subset of size n, then, under the stepping-up coloring χ', the set $J = \{0,1\}^N$ contains no homogeneous subset of size $2n + k - 4$.*

The proof is not very complicated and it can be found, e.g., in [CFP+14] or [GRS90, Sec. 4.7].

Semialgebraic stepping-up. Now let Φ be a d-dimensional k-ary semialgebraic predicate, and let $P = (p_1, \dots, p_N)$ be a point sequence in $\mathbb{R}^d$ indexed by the set $I = [N]$ as above. Let $\chi = \chi_\Phi$ be the coloring of k-tuples of I induced by Φ; that is, for $i_1 < \cdots < i_k \in I$, $\chi(\{i_1, \dots, i_k\})$ is 1 or 0 depending on whether $\Phi(p_{i_1}, \dots, p_{i_k})$ holds or not.

We want to construct a sequence Q in $\mathbb{R}^{d+1}$ indexed by $J = \{0,1\}^N$ and a $(d+1)$-dimensional $(k+1)$-ary semialgebraic predicate Ψ such that the coloring induced by Ψ on $\binom{J}{k+1}$ is exactly the stepping-up coloring χ'. For our construction, we need to assume simple additional properties of Φ and P, which we now introduce.

[3] $\Delta = \{(j,j)\colon j \in J\}$

Let $P = (p_1, \dots, p_N)$ be a sequence of points in $\mathbb{R}^d$. We call a predicate Φ *robust*[4] on P if there is some $\eta > 0$ such that $\Phi(p_{i_1}, \dots, p_{i_k}) \Leftrightarrow \Phi(p'_{i_1}, \dots, p'_{i_k})$ whenever $1 \le i_1 < \dots < i_k \le N$ and $\|p_{i_j} - p'_{i_j}\| \le \eta$ for all $j = 1, 2, \dots, k$.

In defining the new predicate Ψ, we will also need to use the linear ordering of the points of P. We thus say that a binary semialgebraic predicate $\prec$ on $\mathbb{R}^d$ is *order-inducing* for P if $p_i \prec p_j$ iff $i < j$, for $i, j = 1, 2, \dots, N$.

Now we can state our semialgebraic stepping-up lemma.

Proposition 2.7 (Semialgebraic stepping-up)**.** *Let Φ be a d-dimensional k-ary semialgebraic predicate and let $\prec$ be a d-dimensional binary semialgebraic predicate. Then there are a $(d+1)$-dimensional $(k+1)$-ary semialgebraic predicate Ψ and a $(d+1)$-dimensional binary semialgebraic predicate $\prec'$ with the following property.*

Let $P = (p_1, \dots, p_N)$ be a point sequence in $\mathbb{R}^d$ such that $\prec$ is order-inducing on P and both Φ and $\prec$ are robust on P, and let χ_Φ be the coloring of k-tuples of $I = [N]$ induced by Φ. Then there is a point sequence $Q = (q_{\boldsymbol\alpha} : \boldsymbol\alpha \in J = \{0,1\}^N)$ such that $\prec'$ is order-inducing on Q (w.r.t. the lexicographic ordering of J), both Ψ and $\prec'$ are robust on Q, and the coloring χ_Ψ induced on the $(k+1)$-tuples of J by Ψ is the stepping-up coloring for χ_Φ.

Proof. The construction of Q uses a parameter $\varepsilon > 0$, which we assume to be sufficiently small.

For $\boldsymbol\alpha = (\alpha_1, \dots, \alpha_N) \in J$, we set

$$q_{\boldsymbol\alpha} := \sum_{i=1}^{N} \alpha_i \varepsilon^i (1, p_{i,1}, p_{i,2}, \dots, p_{i,d}) \in \mathbb{R}^{d+1}.$$

In particular, the first coordinate of $q_{\boldsymbol\alpha}$ is $\sum_{i=1}^N \alpha_i \varepsilon^i$. Hence, as is easy to check, for ε sufficiently small, the lexicographic ordering of J agrees with the ordering of Q by the first coordinate, and hence we can take the standard ordering in the first coordinate as the required order-inducing (and obviously robust) predicate $\prec'$ on Q.

Next, we define a mapping $\overline{\delta}\colon \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \to \mathbb{R}^d$, which will play the role of the δ from the stepping-up lemma in the geometric setting. For points $x, y \in \mathbb{R}^{d+1}$, we set

$$\overline{\delta}(x,y) := \left(\frac{x_2 - y_2}{x_1 - y_1}, \frac{x_3 - y_3}{x_1 - y_1}, \dots, \frac{x_{d+1} - y_{d+1}}{x_1 - y_1} \right) \in \mathbb{R}^d. \tag{2.2}$$

(Actually, $\overline{\delta}(x,y)$ is undefined for $x_1 = y_1$, but we will use $\overline{\delta}$ only for points with different first coordinates.)

By elementary calculation we can see that for $\boldsymbol\alpha, \boldsymbol\beta \in J$, $\boldsymbol\alpha \ne \boldsymbol\beta$, we have

$$\lim_{\varepsilon \to 0} \overline{\delta}(q_{\boldsymbol\alpha}, q_{\boldsymbol\beta}) = p_{\delta(\boldsymbol\alpha, \boldsymbol\beta)}. \tag{2.3}$$

This allows us to imitate the combinatorial definition (2.1) of the stepping-up coloring by a semialgebraic predicate Ψ. For a $(k+1)$-tuple of points $(x_1, \dots, x_{k+1})$ in $\mathbb{R}^{d+1}$, let us write $\overline{\delta}_\ell := \overline{\delta}(x_\ell, x_{\ell+1})$, and set

$$\Psi(x_1, \dots, x_{k+1}) := \begin{cases} \Phi(\overline{\delta}_1, \dots, \overline{\delta}_k) & \text{if } \overline{\delta}_1 \prec \dots \prec \overline{\delta}_k \\ \Phi(\overline{\delta}_k, \dots, \overline{\delta}_1) & \text{if } \overline{\delta}_1 \succ \dots \succ \overline{\delta}_k \\ \text{true} & \text{if } \overline{\delta}_1 \prec \overline{\delta}_2 \succ \overline{\delta}_3 \\ \text{false} & \text{otherwise.} \end{cases}$$

[4]Conlon et al. [CFP+14] use the term η-deep.

As written, Ψ is not necessarily a semialgebraic predicate, since the definition of $\overline{\delta}$ involves division. However, we can always multiply by the denominators and introduce appropriate conditions; e.g., $\frac{u}{v} < 1$ can be replaced with $(u < v \wedge v > 0) \vee (u > v \wedge v < 0)$, which is equivalent whenever $\frac{u}{v}$ is defined. In this way, we obtain an honest semialgebraic predicate.

It remains to check that Ψ induces the stepping-up coloring on J, which is straightforward using the robustness of Φ and $\prec$ and the limit relation (2.3). Indeed, let us fix $\boldsymbol{\alpha}_1 <_{\text{lex}} \cdots <_{\text{lex}} \boldsymbol{\alpha}_{k+1} \in J$ and write $\overline{\delta}_\ell := \overline{\delta}(q_{\boldsymbol{\alpha}_\ell}, q_{\boldsymbol{\alpha}_{\ell+1}})$ and $\delta_\ell := \delta(\boldsymbol{\alpha}_\ell, \boldsymbol{\alpha}_{\ell+1})$. Then for ε sufficiently small, we have $\overline{\delta}_\ell \prec \overline{\delta}_{\ell+1}$ iff $p_{\delta_\ell} \prec p_{\delta_{\ell+1}}$ (by the robustness of $\prec$) iff $\delta_\ell < \delta_{\ell+1}$ (since $\prec$ is order-inducing on P). Assuming $\overline{\delta}_1 \prec \overline{\delta}_2 \prec \cdots \prec \overline{\delta}_k$, we get that $\Phi(\overline{\delta}_1, \ldots, \overline{\delta}_k)$ iff $\Phi(p_{\delta_1}, \ldots, p_{\delta_k})$, again for all sufficiently small ε; similarly if $\overline{\delta}_1 \succ \overline{\delta}_2 \succ \cdots \succ \overline{\delta}_k$. Therefore, the coloring induced by Ψ on Q is indeed the stepping-up coloring for χ_Φ as claimed.

It remains to verify that Ψ is robust on Q, but this is clear from the robustness of Φ and $\prec$ and the continuity of $\overline{\delta}$ on the subset of $\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$ where it is defined. □

Proof of Theorem 2.3. As announced, we prove the theorem by induction on d.

For the base case $d = 1$, we use a result of Conlon et al. [CFP+14], who construct a 4-ary semialgebraic predicate Φ_1 on $\mathbb{R}^1$ and, for every n, a sequence $P_1 \subset \mathbb{R}$ of length $\operatorname{twr}_3(\Omega(n))$ with no Ψ_1-homogeneous subsequence of length n. It is obvious from their construction that Ψ_1 is robust on P_1 and that $<$, the usual inequality among real numbers, is robust and order-inducing on P_1.

The theorem then follows by a $(d-1)$-fold application of Proposition 2.7 together with the stepping-up lemma (Lemma 2.6). □

2.3 Lower bound for super-order-type

Here we prove Theorem 2.4. Thus, we need to exhibit long point sequences without super-order-type homogeneous subsequences of length n. The construction is almost identical to the one in the previous section, only the base case for $d = 1$ is different. The proof essentially consists in relating super-order-type homogeneity to another property, which we call super-monotonicity; checking that the constructed sequence has no super-monotone subsequences of length n is straightforward.

First, for convenience, we extend the definition of the bivariate function $\overline{\delta}$ from (2.2) in the previous section to an arbitrary number of arguments. Namely, we set $\overline{\delta}(p) = p$ and, for $k \geq 2$,

$$\overline{\delta}(p_1, \ldots, p_{k+1}) := \overline{\delta}(\overline{\delta}(p_1, \ldots, p_k), \overline{\delta}(p_2, \ldots, p_{k+1})).$$

Again, we are going to use $\overline{\delta}$ only with arguments for which it is well-defined.

For points $p, q \in \mathbb{R}^d$, we write $p <_1 q$ if $p_1 < q_1$ (strict inequality in the first coordinate). A point sequence $P = (p_1 \ldots, p_n)$ in $\mathbb{R}^d$ is *super-monotone* if each of the point sequences $(\overline{\delta}(p_1, \ldots, p_j), \ldots, \overline{\delta}(p_{n-j+1}, \ldots, p_n))$ in $\mathbb{R}^{d-j+1}$ is monotone according to $<_1$, $1 \leq j \leq d$.

Here is the key technical result.

Proposition 2.8. *A point sequence $(p_1, \ldots, p_n)$ in $\mathbb{R}^d$ is super-monotone if and only if it is super-order-type homogeneous.*

The proof will be given at the end of this section, after some algebraic lemmas. First we finish the proof of Theorem 2.4, assuming the proposition.

Proof of Theorem 2.4. We will construct a sequence $P_d(n)$ in general position in $\mathbb{R}^d$ of length $\mathrm{twr}_d(n-d)$ and containing no super-order-type homogeneous subsequence of length n.

We proceed by induction on d. The inductive hypothesis will include the assumption that the first coordinate in $P_d(n)$ is strictly increasing.

For $d = 1$ we set $P_1(n) := (1, 2, \dots, n-1)$.

Now we construct $P_{d+1}(n)$ from $P_d(n-1) = (p_1, \dots, p_N)$, using the same construction as in Proposition 2.7. That is, $P_{d+1}(n) = (q_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \{0,1\}^N)$, where the binary vectors $\boldsymbol{\alpha}$ are ordered lexicographically, and where, with $\varepsilon > 0$ sufficiently small,

$$q_{\boldsymbol{\alpha}} := \sum_{i=1}^{N} \alpha_i \varepsilon^i (1, p_{i,1}, p_{i,2}, \dots, p_{i,d}) \in \mathbb{R}^{d+1}, \quad \boldsymbol{\alpha} \in \{0,1\}^N.$$

(The ε is different in each inductive step, and in particular, the one used to construct $P_{d+1}(n)$ from $P_d(n-1)$ is much smaller than the one used to construct $P_d(n-1)$ from $P_{d-1}(n-2)$, etc.) Because of the robustness of the super-order-type condition, we can slightly perturb the points so that they are in general position. As in the previous section, the points of $P_{d+1}(n)$, ordered according to the lexicographic ordering of the indices $\boldsymbol{\alpha}$, have increasing first coordinates (for ε sufficiently small).

Now we assume for contradiction that $P_{d+1}(n)$ contains a super-order-type homogeneous subsequence $S = (s_1, \dots, s_n)$. By Proposition 2.8, S is super-monotone. Thus, setting $t_\ell = \overline{\delta}(s_\ell, s_{\ell+1})$, $\ell = 1, 2, \dots, n-1$, the sequence $T = (t_1, \dots, t_{n-1})$ is super-monotone as well by definition.

By the limit relation (2.3), for $\varepsilon \to 0$, each t_ℓ tends to a point p_{i_ℓ} of $P_d(n-1)$. Moreover, by super-monotonicity, we have $t_1 <_1 \cdots <_1 t_{n-1}$. Hence $p_{i_1} <_1 \cdots <_1 p_{i_{n-1}}$ for sufficiently small ε and therefore, since the first coordinates are increasing in $P_d(n-1)$ by the inductive hypothesis, we have $i_1 < \cdots < i_{n-1}$. Consequently, using Proposition 2.8 again, $(p_{i_1}, \dots, p_{i_{n-1}})$ is a super-order-type homogeneous subsequence of $P_d(n-1)$—a contradiction proving the theorem. □

Algebraic lemmas. It remains to prove Proposition 2.8, and for this, we need to develop some algebraic results.

Given a k-tuple $T = (p_1, \dots, p_k)$ of points in $\mathbb{R}^d$, $1 \le k \le d$, and an index $j \ge k-1$, we put

$$D_j(T) = \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ p_{1,1} & p_{2,1} & \cdots & p_{k,1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1,k-2} & p_{2,k-2} & \cdots & p_{k,k-2} \\ p_{1,j} & p_{2,j} & \cdots & p_{k,j} \end{pmatrix}$$

and

$$\overrightarrow{D}_j(T) = (D_j(T), D_{j+1}(T), \dots, D_d(T)).$$

Let us remark that k is not represented explicitly in the notation, but it can be inferred from the number of arguments of D_j. We also note that $\operatorname{sgn} D_{k-1}(p_1, \dots, p_k)$ is the sign of the k-tuple $\pi_{k-1}(T)$.

Lemma 2.9. *If $A = (p_1, \dots, p_k)$ and $B = (p_2, \dots, p_{k+1})$, then, for $j \geq k$, we have*

$$D_{k-1}(A)D_j(B) - D_{k-1}(B)D_j(A) = D_{k-2}(p_2, \dots, p_k)D_j(p_1, \dots, p_{k+1}).$$

Proof. It is enough to do the case $j = k$ (we have $j \geq k$, and so in the identity of the lemma, the jth coordinates of the p_i appear only in the determinants $D_j(A)$, $D_j(B)$, and $D_j(p_1, \dots, p_{k+1})$). We define the $(k+1) \times (k+1)$ matrix

$$M_{k+1}(p_1, \dots, p_{k+1}) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ p_{1,1} & p_{2,1} & \cdots & p_{k+1,1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1,k} & p_{2,k} & \dots & p_{k+1,k} \end{pmatrix}.$$

All the determinants we are interested in are submatrices of M_{k+1} and they all contain the matrix $M_{k-1} = M_{k-1}(p_2, \dots, p_k)$ associated with $D_{k-2}(p_2, \dots, p_k)$. We can use elementary row and column operations on M_{k+1} to diagonalize M_{k-1} while leaving the determinants fixed, and we can also assume that the entries below M_{k-1}, as well as those to the left and to the right of it, are 0, as is illustrated next:

$$\begin{pmatrix} 1 & \boxed{\begin{matrix} 1 & \dots & 1 \\ p_{2,1} & \cdots & p_{k,1} \\ \vdots & M_{k-1} & \vdots \\ p_{2,k-2} & \cdots & p_{k,k-2} \end{matrix}} & 1 \\ p_{1,1} & & p_{k+1,1} \\ \vdots & & \vdots \\ p_{1,k-2} & & p_{k+1,k-2} \\ p_{1,k-1} & p_{2,k-1} \quad \cdots \quad p_{k+1,k-1} & p_{k+1,k-1} \\ p_{1,k} & p_{2,k} \quad \cdots \quad p_{k+1,k} & p_{k+1,k} \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & \boxed{\begin{matrix} m_1 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & m_{k-2} \end{matrix}} & 0 \\ 0 & & 0 \\ \vdots & & \vdots \\ 0 & & 0 \\ x & 0 \quad \dots \quad 0 & u \\ y & 0 \quad \dots \quad 0 & v \end{pmatrix}.$$

Now we can compute the determinants in the following way:

$$D_k(p_1, \dots, p_{k+1}) = (-1)^{k+1}(xv - yu)\det(M_{k-1}) \qquad D_k(B) = v\det(M_{k-1})$$
$$D_{k-2}(p_2, \dots, p_k) = \det(M_{k-1}) \qquad D_k(A) = (-1)^{k+1}y\det(M_{k-1})$$
$$D_{k-1}(A) = (-1)^{k+1}x\det(M_{k-1}) \qquad D_{k-1}(B) = u\det(M_{k-1}).$$

The lemma follows. □

Lemma 2.10.

$$\overline{\delta}(p_1, \dots, p_k) = \frac{\overrightarrow{D}_k(p_1, \dots, p_k)}{D_{k-1}(p_1, \dots, p_k)}.$$

Proof. The proof goes by induction on k. The cases $k = 1, 2$ are trivial. Assume the lemma is true for k and we have points $p_1, \dots, p_{k+1} \in \mathbb{R}^d$. For simplicity we write $A = (p_1, \dots, p_k)$ and $B = (p_2, \dots, p_{k+1})$. Let $\pi\colon \mathbb{R}^d \to \mathbb{R}^{d-1}$ be the projection on the last $d-1$ coordinates and let $(\cdot)_1$ denote the first coordinate of a vector in $\mathbb{R}^d$. Then we have

$$\begin{aligned} \overline{\delta}(p_1, \dots, p_{k+1}) &= \overline{\delta}(\overline{\delta}(A), \overline{\delta}(B)) \\ &= \frac{\pi(\overline{\delta}(A)) - \pi(\overline{\delta}(B))}{(\overline{\delta}(A))_1 - (\overline{\delta}(B))_1} \\ &= \frac{D_{k-1}(A)\overrightarrow{D}_{k+1}(B) - D_{k-1}(B)\overrightarrow{D}_{k+1}(A)}{D_{k-1}(A)D_k(B) - D_{k-1}(B)D_k(A)}. \end{aligned}$$

The second equality follows from the definition of $\overline{\delta}$ given in (2.2). The last equality follows from the fact that $\pi(\overrightarrow{D}_k) = \overrightarrow{D}_{k+1}$ and by clearing the denominators. To finish the proof we use Lemma 2.9 on the denominator and each coordinate of the numerator. □

Proof of Proposition 2.8. We generalize the notions of super-monotonicity and super-order-type homogeneity as follows. We say that a point sequence $(p_1 \ldots, p_n)$ is k-*monotone* if for all $j \leq k$ the point sequence $(\overline{\delta}(p_1, \ldots, p_j), \ldots, \overline{\delta}(p_{n-j+1}, \ldots, p_n))$ is monotone according to $<_1$. We say that $(p_1 \ldots, p_n)$ is k*-order-type homogeneous* if for all $j \leq k$ the sequence of projections $(\pi_j(p_1) \ldots, \pi_j(p_n))$ in $\mathbb{R}^j$ is order-type homogeneous.

By induction on k, we prove that a point sequence $(p_1, \ldots, p_n)$ in $\mathbb{R}^d$ is k-monotone if and only if it is k-order-type homogeneous; for $k = d$ this is the statement of the proposition.

The cases $k = 1, 2$ are trivial. So we assume that the claim is true up to some k and we are given a sequence of n points. We may assume that this sequence is k-monotone, and hence also k-order-type homogeneous. Then we only need to show that $(\overline{\delta}(p_1, \ldots, p_{k+1}), \ldots, \overline{\delta}(p_{n-k}, \ldots, p_n))$ is order-type homogeneous if and only if $(\pi_{k+1}(p_1), \ldots, \pi_{k+1}(p_n))$ is monotone according to $<_1$.

Let $(q_1, \ldots, q_{k+2})$ be a $(k+2)$-point subsequence of $(p_1, \ldots, p_n)$. By Lemma 2.10, the condition $\overline{\delta}(q_1, \ldots, q_{k+1}) <_1 \overline{\delta}(q_2, \ldots, q_{k+2})$ is equivalent to

$$\frac{D_{k+1}(q_2, \ldots, q_{k+2})}{D_k(q_2, \ldots, q_{k+2})} - \frac{D_{k+1}(q_1, \ldots, q_{k+1})}{D_k(q_1, \ldots, q_{k+1})} > 0. \tag{2.4}$$

Since $(q_1, \ldots, q_{k+2})$ is k-order-type homogeneous, we have

$$D_k(q_1, \ldots, q_{k+1}) D_k(q_2, \ldots, q_{k+2}) > 0,$$

and therefore, (2.4) is equivalent to

$$D_{k+1}(q_2, \ldots, q_{k+2}) D_k(q_1, \ldots, q_{k+1}) - D_{k+1}(q_1, \ldots, q_{k+1}) D_k(q_2, \ldots, q_{k+2}) > 0.$$

By Lemma 2.9 this is just

$$D_{k-1}(q_2 \ldots, q_{k+1}) D_{k+1}(q_1 \ldots, q_{k+2}) > 0.$$

Since our sequence is also $(k-1)$-order-type homogeneous, the numbers

$$D_{k-1}(p_1 \ldots, p_k),\ D_{k-1}(p_2 \ldots, p_{k+1}),\ \ldots,\ D_{k-1}(p_{n-k+1} \ldots, p_n)$$

have the same sign and therefore the numbers

$$D_{k+1}(p_1 \ldots, p_{k+2}),\ D_{k+1}(p_2 \ldots, p_{k+3}),\ \ldots,\ D_{k+1}(p_{n-k-1} \ldots, p_n)$$

also have the same sign. This is precisely the condition needed for the sequence to be $(k+1)$-order-type homogeneous. □

Chapter 3

Reptile simplices in $\mathbb{R}^3$ and $\mathbb{R}^4$

3.1 Introduction

A *tiling* of a closed set X in $\mathbb{R}^d$ (or in the unit sphere S^d) is a locally finite decomposition $X = \bigcup_{i \in I} X_i$ into closed sets with nonempty and pairwise disjoint interiors. The sets X_i are called *tiles*. If X has a tiling where all the tiles are congruent to a set T, we say that T *tiles* X, or, that X *can be tiled with (*$|I|$ *copies of)* T. We emphasize that congruence includes mirror symmetries.

A closed set $X \subset \mathbb{R}^d$ with nonempty interior is called a *k-reptile* (or a *k-reptile set*) if X can be tiled with k mutually congruent copies of a set similar to X.

It is easy to see that whenever S is a d-dimensional k-reptile set, then S is *space-filling*, that is, the space $\mathbb{R}^d$ can be tiled with S: indeed, using the tiling of S by its smaller copies as a pattern, one can inductively tile larger and larger similar copies of S. On the other hand, it is a simple exercise to find space-filling polytopes or polygons that are not k-reptiles for any $k \geq 2$.

Clearly, every triangle tiles $\mathbb{R}^2$. Moreover, every triangle T is a k-reptile for $k = m^2$, since T can be tiled in a regular way with m^2 congruent tiles, each positively or negatively homothetic to T. See Snover et al. [SWW91] for an illustration.

In this chapter we study the existence of k-reptile *simplices* in $\mathbb{R}^d$, especially for $d = 3$ and $d = 4$.

Space-filling simplices. The question of characterizing the tetrahedra that tile $\mathbb{R}^3$ is still open and apparently rather difficult. The first systematic study of space-filling tetrahedra was made by Sommerville, who discovered a list of exactly four tilings (up to isometry and rescaling), but he assumed that all tiles are *properly congruent* (that is, congruent by an orientation-preserving isometry) and meet face-to-face [Som23]. Edmonds [Edm07] noticed a gap in Sommerville's proof and by completing the analysis, he confirmed that Sommerville's classification of proper, face-to-face tilings is complete. Baumgartner [Bau71] found three of Sommerville's tetrahedra and one new tetrahedron that admits a non-proper face-to-face tiling (and also a proper non face-to-face tiling [Gol74]). Goldberg [Gol74] described three families of proper (generally not face-to-face) tilings, obtained by partitioning a triangular prism. In fact, Goldberg's first family was found by Sommerville [Som23] before, but he selected only special cases with a certain symmetry. Goldberg's first family also coincides with the family of simplices found by Hill [Hil95], whose aim was to classify *rectifiable* simplices, that is, simplices that can be cut by straight cuts into finitely many pieces that can be rearranged to form a cube. The simplices in Goldberg's second and third families are obtained from the simplices in the first family by splitting into two congruent halves. According to Senechal's survey [Sen81], no other space-filling tetrahedra are known.

For $d \geq 3$, Debrunner [Deb85] constructed $\lfloor d/2 \rfloor + 2$ one-parameter families and several special types of d-dimensional simplices that tile $\mathbb{R}^d$. Smith [Smi03] generalized Goldberg's construction and using Debrunner's ideas, he obtained $(\lfloor d/2 \rfloor + 2)\phi(d)/2$ one-parameter families of space-filling d-dimensional simplices; here $\phi(d)$ is the Euler's totient function. It is not known whether for some $d \geq 3$ there is a space-filling simplex with all dihedral angles acute or a two-parameter family of space-filling simplices [Smi03].

Hilbert's problems. Two Hilbert's problems are related to tilings of the Euclidean space. The second part of Hilbert's 18th problem asks whether there exists a polyhedron that tiles the 3-dimensional Euclidean space but does not admit an isohedral (tile-transitive) tiling. The first such tile in three dimensions was found by Reinhardt [Rei28]. Later Heesch [Hee35] found a planar anisohedral nonconvex polygon and Kershner [Ker68] found an anisohedral convex pentagon. Hilbert's 18th problem was discussed in detail by Milnor [Mil76]. See also the survey by Grünbaum and Shepard [GS80] for a discussion of this problem and related questions. While iterated tilings of the space using tilings of some k-reptiles as a pattern may be highly irregular, it is an interesting question whether there is an anisohedral k-reptile polytope or polygon. Vince [Vin95, Question 2] asked whether there is a k-reptile that admits no periodic tiling.

The third Hilbert's problem asks whether two tetrahedra with equal bases and altitudes are *equidecomposable*, that is, whether one can cut one tetrahedron into finitely many polytopes and reassemble them to form the second tetrahedron. A positive answer would provide an elementary proof of the formula for the volume of the tetrahedron. However, Dehn [Deh01] answered the question in the negative, by introducing an algebraic invariant for equidecomposability. See [Do06] for an elementary exposition or [AZ10, Chapter 9], [Ben07] for alternative proofs. Debrunner [Deb80] proved that every polytope that tiles $\mathbb{R}^d$ has its *codimension 2 Dehn's invariant* equal to zero. Lagarias and Moews [LM95a, LM95b] showed that, more generally, every polytope that tiles $\mathbb{R}^d$ has its *classical total Euclidean Dehn's invariant* equal to zero. By the results of Sydler [Syd65] (see also [Jes68]) and Jessen [Jes72], this implies that for $d = 3$ and $d = 4$, respectively, every polytope that tiles $\mathbb{R}^d$ is equidecomposable with a cube [Deb80, LM95a, LM95b]. In particular, these properties are necessary for every k-reptile simplex.

Reptiles and other animals. Motivated by classical puzzles that require splitting a given figure into a given number of congruent *replicas* of the original figure, Langford [Lan40] initiated a systematic study of planar k-reptiles. Golomb [Gol64] introduced the term *replicating figure of order k*, shortly *a rep-k*, and described several more examples, including disconnected or totally disconnected fractal tiles. See also Gardner's [Gar91] short survey. Extending the theory of self-similar sets and fractals, Bandt [Ban91] described a general construction of infinitely many self-similar k-reptiles, including several species of dragons, which are examples of *disk-like* (that is, homeomorphic to a disk) reptiles. Gelbrich [Gel94] proved that for every k, there are only finitely many planar disk-like crystallographic (isohedral) k-reptiles. See Gelbrich and Giesche [GG98] for illustrations of several such 7-reptiles, such as sea horses or salamanders.

***k*-reptile simplices.** In recent years the subject of tilings has received a certain impulse from computer graphics and other computer applications. In fact, our original motivation for studying simplices that are k-reptiles comes from a problem of probabilistic marking of Internet packets for IP traceback [Adl02, AEM05]. See [Mat05] for a brief summary of the ideas of this method. For this application, it would be interesting to find a d-dimensional simplex that is a k-reptile with k as small as possible.

For dimension 2 there are several possible types of k-reptile triangles, and they have been completely classified by Snover et al. [SWW91]. In particular, k-reptile triangles exist for all k of the form $a^2 + b^2$, a^2 or $3a^2$ for nonzero integers a, b. In contrast, for $d \geq 3$, reptile simplices seem to be much more rare. The only known constructions of higher-dimensional k-reptile simplices have $k = m^d$. The best known examples are the *Hill simplices* (or the *Hadwiger–Hill simplices*) [Deb85, Had51, Hil95]. A d-dimensional Hill simplex is the convex hull of vectors 0, b_1, $b_1 + b_2$, $\dots$, $b_1 + \cdots + b_d$, where $b_1, b_2, \dots, b_d$ are vectors of equal length such that the angle between every two of them is the same and lies in the interval $(0, \frac{\pi}{2} + \arcsin \frac{1}{d-1})$.

Hertel [Her00] proved that a 3-dimensional simplex is an m^3-reptile using a "standard" way of dissection (which we will not define here) if and only if it is a Hill simplex. He conjectured that Hill simplices are the only 3-dimensional reptile simplices. Herman Haverkort recently pointed us to an example of a k-reptile tetrahedron by Liu and Joe [LJ94] which is not Hill, and thus contradicts Hertel's conjecture. In fact, except for the one-parameter family of Hill tetrahedra, two other space-filling tetrahedra described by Sommerville [Som23] and Goldberg [Gol74] are also k-reptiles for every $k = m^3$. Both these tetrahedra tile the right-angled Hill tetrahedron, and their tilings are based on the barycentric subdivision of the cube. Maehara [Mae13] described this construction for $k = 2^d$. It is easy to see that the lattice tiling of $\mathbb{R}^d$ by barycentrically subdivided unit cubes can be obtained by cutting the space with hyperplanes $x_i = n/2$, $x_i + x_j = n$, $x_i - x_j = n$, for every $i, j \in [d], i \neq j$ and $n \in \mathbb{Z}$. This tiling contains, for every m, a tiling of an m times scaled copy of the right-angled Hill simplex. Similarly, by removing the hyperplanes $x_i = (2n+1)/2$ from the cutting, we obtain a tiling of $\mathbb{R}^d$ with tiles that are made of two copies of the Hill simplex. Again, this tiling contains a tiling of an m times scaled copy of each tile.

Matoušek [Mat05] showed that there are no 2-reptile simplices of dimension 3 or larger. For dimension $d = 3$, Matoušek and the author [MS11] proved the following theorem.

Theorem 3.1 ([MS11])**.** *In $\mathbb{R}^3$, k-reptile simplices (tetrahedra) exist only for k of the form m^3, where m is a positive integer.*

We give a new simple proof of Theorem 3.1 in Section 3.3.

Matoušek and the author [MS11] conjectured that for $d \geq 3$, a d-dimensional k-reptile simplex exists only for k of the form m^d for some positive integer m. We prove a weaker version of this conjecture for four-dimensional simplices.

Theorem 3.2. *Four-dimensional k-reptile simplices can exist only for k of the form m^2, where m is a positive integer.*

Four-dimensional Hill simplices are examples of k-reptile simplices for $k = m^4$. Whether there exists a four-dimensional m^2-reptile simplex for m non-square remains an open question.

New ingredients. As an important tool we use Debrunner's lemma [Deb85], which connects the symmetries of a d-simplex with the symmetries of its Coxeter diagram (which represents the "arrangement" of the dihedral angles). This lemma allows us to substantially simplify the proof of Theorem 3.1 and enables us to step one dimension up and prove Theorem 3.2, which seemed unmanageable before.

In the proof of Theorem 3.2 we encounter the problem of tiling spherical triangles by congruent triangular tiles, which might be of independent interest. A related question, a classification of edge-to-edge tilings of the sphere by congruent triangles, has been completely solved by Agaoka and Ueno [UA02].

3.2 Basic notions and facts about simplices and group actions

3.2.1 Angles in simplices and Coxeter diagrams

Given a d-dimensional simplex S with vertices $v_1, \ldots, v_{d+1}$, let F_i be the facet opposite to v_i. If $\alpha_{i,j}$ is the angle between the normals of F_i and F_j pointing outward, then the *dihedral angle* $\beta_{i,j}$ is defined as $\pi - \alpha_{i,j}$. By an *internal angle* φ *at the point* x *of* S, where x is on the boundary of S, we mean the set $\mathbb{S}^{d-1}(x, \varepsilon) \cap S$, where $\mathbb{S}^{d-1}(x, \varepsilon)$ denotes the $(d-1)$-dimensional sphere with radius ε centered at x, where $\varepsilon > 0$ is small enough (so that $\mathbb{S}^{d-1}(x, \varepsilon)$ does not meet the facets not containing x). An *edge-angle* of S is the internal $(d-1)$-dimensional angle at an interior point of an edge of S and can be represented by a $(d-2)$-dimensional spherical simplex. Indeed, select an interior point x of the edge e and consider a hyperplane h orthogonal to e and containing x. The edge-angle incident to e can be represented as the intersection $h \cap S \cap \mathbb{S}^{d-1}(x, \varepsilon)$. This intersection is clearly $(d-2)$-dimensional and forms a spherical simplex.

From now on we normalize all edge-angles, that is, we consider them as subsets of the $(d-2)$-dimensional unit sphere.

The *Coxeter diagram* of S is a graph $c(S)$ with labeled edges such that the vertices of $c(S)$ represent the facets of S and for every pair of facets F_i and F_j, there is an edge $e_{i,j}$ labeled by the dihedral angle $\beta_{i,j}$. Note that our labeling differs from the traditional one, where the edge corresponding to a dihedral angle π/p is labeled by p and the label 3 is omitted. Debrunner [Deb85] labels the edge corresponding to a dihedral angle $\beta_{i,j}$ by $\cos \beta_{i,j}$.

Observation 3.3. *The edge-angles of a four-dimensional simplex S can be represented by spherical triangles, whose angles are dihedral angles in S. Therefore, an edge-angle in S represented by a spherical triangle with angles α, β, γ corresponds to a triangle in the Coxeter diagram with edges labeled by α, β, γ.* □

The following important lemma is by Debrunner [Deb85, Lemma 1]. Here the *symmetries of S* are Euclidean isometries, and the *symmetries of c(S)* are graph automorphisms preserving the labels of edges.

Lemma 3.4 (**Debrunner's lemma** [Deb85])**.** *Let S be a d-dimensional simplex. The symmetries of S are in one-to-one correspondence with the symmetries of its Coxeter diagram $c(S)$, in the following sense: each symmetry φ of S induces a symmetry Φ of $c(S)$ so that $\varphi(v_i) = v_j \Leftrightarrow \Phi(F_i) = F_j$, and vice versa.*

3.2.2 Existence of simplices with given dihedral angles

The following elegant property of $\binom{d+1}{2}$-tuples of dihedral angles is by Fiedler [Fie54]. A proof in English can be found in [MS11].

Theorem 3.5 (**Fiedler's theorem** [Fie54])**.** *Let $\beta_{i,j}$, $i, j = 1, 2, \dots, d+1$, be the dihedral angles of some d-dimensional simplex, let $\beta_{i,i} = \pi$ for convenience, and let A be the $(d+1)\times(d+1)$ matrix with $a_{i,j} := \cos\beta_{i,j}$ for all i, j. Then A is negative semidefinite of rank d, and the (1-dimensional) kernel of A is generated by a vector $z \in \mathbb{R}^{d+1}$ with all components strictly positive.*

In our proof of Theorem 3.2 we use only the fact that the matrix A defined in Theorem 3.5 is singular (it is a $(d+1)\times(d+1)$ matrix of rank d).

3.2.3 Group actions

An *action* φ of a group G on a set M is a homomorphism from G to the symmetric group $\mathrm{Sym}(M)$ of M, where symmetric group $\mathrm{Sym}(M)$ is the group of all permutations of M. We say that an action φ of G on M is *faithful* if its kernel is trivial. In other words, φ is faithful if for every $g \neq \mathrm{id}$ there exists an element $m \in M$ with $\varphi(g)(m) \neq m$. It is usual to omit φ and write just gm instead of $\varphi(g)(m)$.

The set $Gm := \{gm : g \in G\}$ is called *orbit* of the element m under the action of G. It is obvious that the set of orbits forms a partition of M. The following well-known lemma counts the number of distinct orbits.

Lemma 3.6 (**Burnside's lemma** [Bur97])**.** *Let M be a finite set and G a finite group acting on M via $m \mapsto gm$. Let X_g be the number of elements of M which are fixed by g, that is, which satisfy the identity $m = gm$. Then the action of G on M has exactly $\frac{1}{|G|}\sum_{g\in G} X_g$ orbits.*

We will need the following lemma:

Lemma 3.7. *Let M be a finite set and G a finite group acting on M nontrivially and faithfully via $m \mapsto gm$. Then G also acts on the (unordered) pairs $\{m, n\} \in \binom{M}{2}$ via $g\{m, n\} = \{gm, gn\}$ and the action of G on pairs has at most $\binom{|M|}{2} - |M| + 2$ orbits. Moreover, the bound is tight and it is achieved if the image of G under the action is generated by a single transposition.*

Proof. Let o_1, o_2 denote the number of orbits of G acting on M, $\binom{M}{2}$, respectively. The group acts on M nontrivially, thus $o_1 \leq |M| - 1$.

Let X_g be the number of elements of M which are fixed by g, that is, elements m such that $m = gm$. We show that the number of elements of $\binom{M}{2}$ which are fixed by g is $\binom{X_g}{2} + \frac{1}{2}(X_{g^2} - X_g)$. Indeed, there are two possibilities for stabilizing the pair $\{m, n\}$:

1. $gm = m$ and $gn = n$; which provides $\binom{X_g}{2}$ fixed elements.

2. $gm = n$, $gn = m$, what can be rewritten as $ggn = n$ and $gn \neq n$. Thus there are $(X_{g^2} - X_g)$ ordered pairs (u, v), $u \neq v$ satisfying $g(u, v) = (v, u)$ and hence $\frac{1}{2}(X_{g^2} - X_g)$ fixed elements $\{u, v\}$.

By Burnside's lemma, we have

$$o_2 = \frac{1}{|G|} \sum_{g \in G} \left(\binom{X_g}{2} + \frac{1}{2}(X_{g^2} - X_g) \right). \tag{3.1}$$

In order to bound (3.1), we need to bound $\sum X_g^2$ in terms of $\sum X_g$:

$$\sum_{g \in G} X_g^2 \leq (|M| - 2) \sum_{g \in G} X_g + 2|M|. \tag{3.2}$$

Indeed, the action is faithful and nontrivial, hence $X_{\mathrm{id}} = |M|$ and $X_g \leq |M| - 2$ otherwise. Using $\sum_{g \neq \mathrm{id}} X_g^2 \leq (|M| - 2) \sum_{g \neq \mathrm{id}} X_g$, the bound (3.2) follows.

Plugging (3.2) into (3.1) and using $X_{g^2} \leq |M|$ we have

$$\begin{aligned} o_2 &= \frac{1}{2|G|} \sum_{g \in G} X_g^2 + \frac{1}{2|G|} \sum_{g \in G} X_{g^2} - \frac{1}{|G|} \sum_{g \in G} X_g \\ &\leq \frac{|M|}{2|G|} \sum_{g \in G} X_g - \frac{2}{|G|} \sum_{g \in G} X_g + \frac{|M|}{|G|} + \frac{|M|}{2}. \end{aligned}$$

Using $|G| \geq 2$, combination of Burnside's lemma $\frac{1}{|G|} \sum X_g = o_1$ and the fact $o_1 \leq |M| - 1$, we get the desired bound:

$$o_2 \leq \frac{(|M| - 4)(|M| - 1)}{2} + |M| = \binom{|M|}{2} - |M| + 2.$$

It remains to show the last part of the statement. But this is clear, since a single transposition swaps $|M| - 2$ pairs of edges. □

3.3 A simple proof of Theorem 3.1

We proceed as in the original proof [MS11], but instead of using the theory of scissors congruence, Jahnel's theorem about values of rational angles and Fiedler's theorem, we only use Debrunner's lemma (Lemma 3.4).

Assume for contradiction that S is a k-reptile tetrahedron where k is not a third power of a positive integer. A dihedral angle α is called *indivisible* if it cannot be written as a linear combination of other dihedral angles in S with nonnegative integer coefficients.

The following lemmas are proved in [MS11].

Lemma 3.8. [MS11, Lemma 3.1] *If α is an indivisible dihedral angle in S, then the edges of S with dihedral angle α have at least three different lengths.*

Lemma 3.8 is analogous to Lemma 3.10, which we prove in the next section.

Lemma 3.9. [MS11, Lemma 3.3] *One of the following two possibilities occur:*

(i) *All the dihedral angles of S are integer multiples of the minimal dihedral angle α, which has the form $\frac{\pi}{n}$ for an integer $n \geq 3$.*

(ii) *There are exactly two distinct dihedral angles β_1 and β_2, each of them occurring three times in S.*

First we exclude case (ii) of Lemma 3.9. If S has two distinct dihedral angles $\beta_1 \neq \beta_2$, each occurring at three edges, then they can be placed in S in two essentially different ways; see Figure 3.1. In both cases, for each $i \in \{1, 2\}$, the Coxeter diagram of S has at least one nontrivial symmetry which swaps two distinct edges with label β_i. By Debrunner's lemma, the corresponding symmetry of S swaps two distinct edges with dihedral angle β_i, which thus have the same length. But then the edges with dihedral angle β_i have at most two different lengths and this contradicts Lemma 3.8, since the smaller of the two angles β_1, β_2 is indivisible.

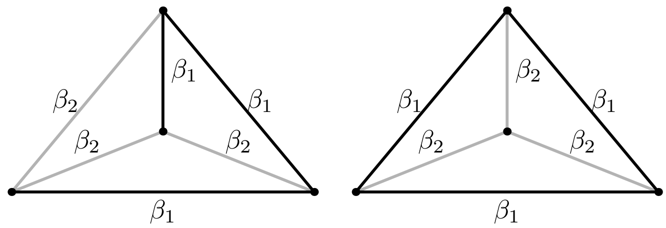


Figure 3.1: Two possible configurations of two dihedral angles.

Now we exclude case (i) of Lemma 3.9. Call the edges of S (and of $c(S)$) with dihedral angle α the *α-edges*. Since there are at least three α-edges in S, there is a vertex v of S where two α-edges meet. Let β be the dihedral angle of the third edge incident to v (possibly β can be equal to α). In the proof of Lemma 3.5 in [MS11] it was shown that $\beta = \pi - \alpha$. The meaning for Coxeter diagrams is that whenever two α-edges meet in $c(S)$, then β is the label of the edge forming triangle with those two α-edes.

Now we distinguish several cases depending on the subgraph H_α of $c(S)$ formed by the α-edges.

- H_α contains three edges incident to a common vertex (which correspond to a triangle in S). Then all the other edges must have the angle β and we get the configuration as in Figure 3.1 (right), which we excluded earlier.

- H_α contains a triangle. Then $\beta = \alpha$, and thus $\alpha = \frac{\pi}{2}$, which contradicts the condition $n \geq 3$ from Lemma 3.9(i).

- H_α contains a path of length three. Then two edges have the angle β and the remaining edge has some angle γ (possibly γ can be equal to α). See Figure 3.2 (left). The symmetric group of the resulting Coxeter diagram always contains $\mathbb{Z}_2$ as a subgroup which corresponds to the involution swapping two α-edges. Unless $\gamma = \alpha$, there are, by Debrunner's lemma, only two α-edge lengths; a contradiction with Lemma 3.8. For $\gamma = \alpha$ the Coxeter diagram has a dihedral symmetry group, D_4, acting transitively on the α-edges, see Figure 3.2 (right). This again contradicts Lemma 3.8, since by Debrunner's lemma, all the α-edges have the same length.

We obtained a contradiction in each of the cases, hence the proof of Theorem 3.1 is finished.

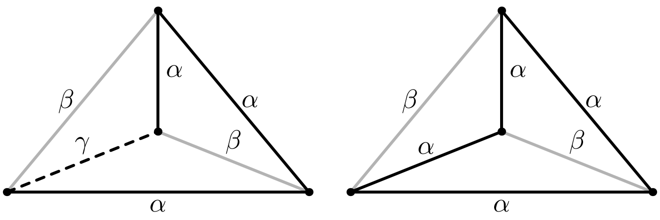


Figure 3.2: The α-edges form a path (left) or a four-cycle (right) in $c(S)$.

3.4 The proof of Theorem 3.2

The method of the proof is similar to the three-dimensional case [MS11].

Assume for contradiction that S is a four-dimensional k-reptile simplex where k is not a square of a positive integer. Let $S_1, \dots, S_k$ be mutually congruent simplices similar to S that tile S. Then each S_i has volume k-times smaller than S, and thus S_i is scaled by the ratio $\rho := k^{-1/4}$ compared to S. For k non-square, ρ is an irrational number of algebraic degree 4 over $\mathbb{Q}$.

Similarly to [MS11] we define an *indivisible* edge-angle (spherical triangle) as a spherical triangle which cannot be tiled with smaller spherical triangles representing the other edge-angles of S or their mirror images. Clearly, the edge-angle with the smallest spherical area is indivisible. In the following we always consider a spherical triangle and its mirror image as the same spherical triangle.

Lemma 3.10. *If $\mathcal{T}_0$ is an indivisible edge-angle in S, then the edges of S with edge-angle $\mathcal{T}_0$ have at least four different lengths (and in particular, there are at least four such edges).*

Proof. The proof is basically the same as for indivisible dihedral angles in tetrahedra [MS11, Lemma 3.1]. Let e be an edge with edge-angle $\mathcal{T}_0$. Every point of e belongs to an edge of some of the smaller simplices S_i. Since $\mathcal{T}_0$ is indivisible, we get that e is tiled by edges of the simplices S_i and each of these edges has edge-angle $\mathcal{T}_0$.

Assume for contradiction that there are at most three edges with edge-angle $\mathcal{T}_0$, with lengths x_1, x_2, x_3. Then the edge of length x_1 is tiled by edges with lengths ρx_1, ρx_2 and ρx_3, and similarly for the edges of lengths x_2 and x_3. In other words, there are nonnegative integers n_{ij}, $i, j = 1, 2, 3$, such that

$$\begin{aligned}
n_{11}\rho x_1 + n_{12}\rho x_2 + n_{13}\rho x_3 &= x_1,\\
n_{21}\rho x_1 + n_{22}\rho x_2 + n_{23}\rho x_3 &= x_2,\\
n_{21}\rho x_1 + n_{22}\rho x_2 + n_{33}\rho x_3 &= x_3.
\end{aligned}$$

This can be rewritten as $\rho\mathbf{A}\mathbf{x} = \mathbf{x}$, where $\mathbf{x} = (x_1, x_2, x_3)^\top$ and $\mathbf{A}$ is the corresponding matrix. Since we assume $\mathbf{x}$ is nonzero, we immediately see that $1/\rho$ is an eigenvalue of $\mathbf{A}$. Since the characteristic polynomial is always nonzero and of degree three, we get a contradiction with $1/\rho$ (and hence also ρ) having algebraic degree 4. □

Since S has 10 edges, Lemma 3.10 implies that there are at most two indivisible edge-angles.

The strategy of the proof is now the following. First we exclude the case of two indivisible edge-angles, using only elementary combinatorial arguments and Debrunner's lemma. Then we consider the case of one indivisible edge-angle. Here we need more involved arguments: we study tilings of spherical triangles with copies of a single spherical triangle and use various observations from spherical geometry and also Fiedler's theorem (Theorem 3.5).

3.4.1 Two indivisible edge-angles

First, we prove an elementary observation about symmetries of the simplex S and its Coxeter diagram.

Lemma 3.11. *If $c(S)$ has a nontrivial symmetry, then the edges of S have at most seven orbits under the action of the symmetry group of S.*

Proof. Let M be the set of vertices of S. By Debrunner's lemma, the symmetry groups of S and $c(S)$ are isomorphic. In particular, S has a nontrivial symmetry group $\Phi \subseteq \mathrm{Sym}(M)$. The inclusion of Φ into $\mathrm{Sym}(M)$ is obviously a faithful action, so we may use Lemma 3.7 for the set M and the group Φ. We immediately get that there are at most seven orbits.

□

Corollary 3.12. *If S has two distinct indivisible edge-angles, then the symmetry group of S (and of $c(S)$) is trivial.*

Proof. By Lemma 3.10, S has at least 4 edges of different lengths for each of the two edge-angles. In particular, no symmetry can identify any two of these 8 edges and so the symmetry group of S induces at least 8 orbits. Therefore it is trivial by Lemma 3.11. □

Now assume for contradiction that S has two indivisible edge-angles $\mathcal{T}_1$ and $\mathcal{T}_2$. Let T_1 and T_2 be the corresponding triangles in $c(S)$. By Lemma 3.10, each of T_1, T_2 occurs at least four times in $c(S)$.

We say that two edges of $c(S)$ are of the same *edge-type* if they have equal labels; that is, they represent equal dihedral angles. An edge of type α is also called an *α-edge*. A triangle T of $c(S)$ with edges of types α, β, γ is called an *$(\alpha\beta\gamma)$-triangle* and we write $T = (\alpha\beta\gamma)$.

Observation 3.13. *Every edge of $c(S)$ belongs to a copy of the triangle T_1 or T_2. Moreover, every edge-type of T_1 and T_2 occurs at least twice in $c(S)$. Consequently, T_1 and T_2 have at least one common edge-type.*

Proof. The first part follows from the fact that every edge of $c(S)$ is contained in three triangles and at least eight of the ten triangles of $c(S)$ are copies of T_1 or T_2. The second claim follows again from the fact that every edge of $c(S)$ is common to only three triangles. □

Observation 3.14. *An edge-type common to both triangles T_1, T_2 occurs at least four times in $c(S)$. Similarly, an edge-type occurring twice in T_1 (or T_2) occurs at least four times in $c(S)$.*

Case	type of T_1	type of T_2	α-edges	β-edges	γ-edges	δ-edges
(1)	$(\alpha\alpha\beta)$	$(\alpha\gamma\delta)$	4	2	2	2
(2)	$(\alpha\alpha\alpha)$	$(\alpha\beta\gamma)$	≤ 6	≥ 2	≥ 2	0
(3)	$(\alpha\alpha\beta)$	$(\alpha\alpha\gamma)$	≤ 6	≥ 2	≥ 2	0
(4)	$(\alpha\alpha\beta)$	$(\alpha\gamma\gamma)$	4	2	4	0
(5)	$(\alpha\alpha\beta)$	$(\alpha\beta\gamma)$	4	4	2	0

Table 3.1: Types of T_1 and T_2 triangles and the corresponding number of edges

Proof. Let α be an edge-type common to both T_1 and T_2 and suppose that each of T_1, T_2 has just one α-edge. There are at least eight triangles with an α-edge in $c(S)$, therefore $c(S)$ has at least three α-edges. But if there are just three α-edges, then some two of them share a vertex (and hence a triangle). Therefore there are at most seven triangles in $c(S)$ with exactly one α-edge.

If T_1 has at least two α-edges, then there are at least four pairs of α-edges in $c(S)$, hence at least four α-edges. □

By Observations 3.13 and 3.14, the edges of $c(S)$ have at most four types in total, since the common edge-type of T_1 and T_2 occurs four times and every other edge-type occurs at least twice. From these observations it also follows that if there are four different edge-types, then three of them, β, γ, δ, appear just once in T_1 or T_2 and the remaining one, α, common to T_1 and T_2, appears twice in T_1 or twice in T_2. Similarly if there are three different edge-types, then one of them appears at least three times together in T_1 and T_2 (counted with multiplicity).

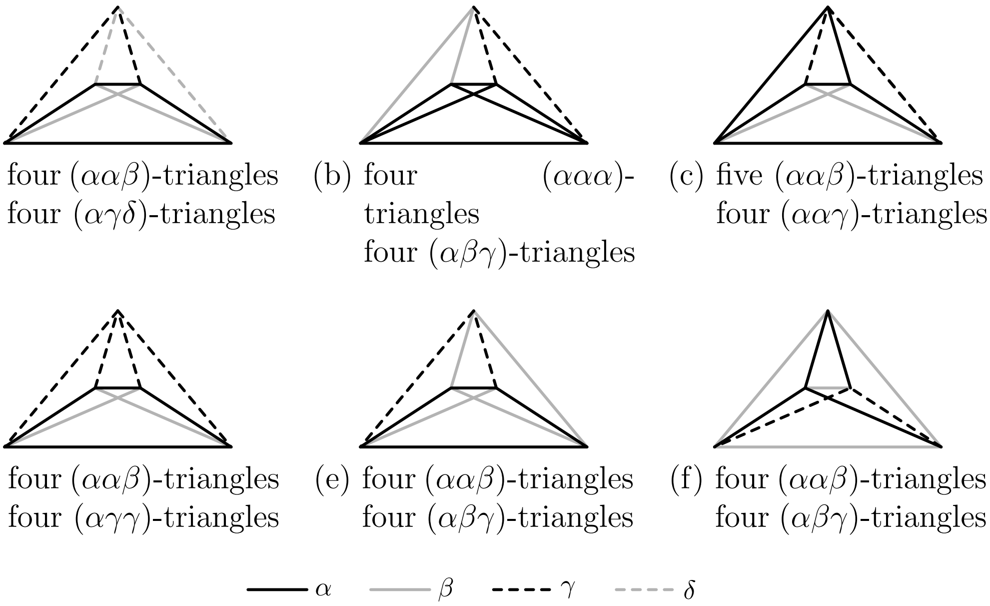
If there are just two different edge-types in $c(S)$, then $c(S)$ has a non-trivial symmetry, which follows from the fact that every graph on five vertices has a nontrivial automorphism. But this contradicts Corollary 3.12.

Thus there are three or four different edge-types in $c(S)$ and we have five essentially different cases for the types of T_1 and T_2 (where $\alpha, \beta, \gamma, \delta$ denote pairwise different angles), see Table 3.1. Moreover, we also have bounds on the number of α, β and γ-edges and for cases (1), (4), (5) we even know their exact number.

In case (1), since there are just two β-edges, some two $(\alpha\alpha\beta)$-triangles in $c(S)$ share a β-edge. This means that the α-edges form a four-cycle. Further it follows that both diagonals of the four-cycle are β-edges and that the fifth vertex of $c(S)$ is joined by γ-edges to two opposite vertices of the four-cycle and by δ-edges to the other pair of opposite vertices; see Figure 3.3a). This diagram has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, which contradicts Corollary 3.12.

Now consider case (2). Since a graph with five edges has at most two triangles, there are at least six α-edges in $c(S)$. Consequently, there are exactly six α-edges, two β-edges and two γ-edges in $c(S)$. Since K_4 is the only graph with six edges and four triangles, the α-edges form a K_4 subgraph in $c(S)$, with two vertices joined by a β-edge and two by a γ-edge to the remaining vertex of $c(S)$; see Figure 3.3b). This diagram has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, in contradiction with Corollary 3.12.

In case (3), the α-edges form a subgraph H_α with at least eight induced paths of length 2 hence H_α has at least five edges. By Observation 3.13, H_α has at most six edges. As in case (1), H_α contains a four-cycle. Adding one more edge does not produce eight induced paths of length 2, therefore H_α has exactly six edges and it is isomorphic to $K_{2,3}$. The remaining edges form a disjoint union of an edge and a triangle and



(a) four ($\alpha\alpha\beta$)-triangles four ($\alpha\gamma\delta$)-triangles

(b) four ($\alpha\alpha\alpha$)-triangles four ($\alpha\beta\gamma$)-triangles

(c) five ($\alpha\alpha\beta$)-triangles four ($\alpha\alpha\gamma$)-triangles

(d) four ($\alpha\alpha\beta$)-triangles four ($\alpha\gamma\gamma$)-triangles

(e) four ($\alpha\alpha\beta$)-triangles four ($\alpha\beta\gamma$)-triangles

(f) four ($\alpha\alpha\beta$)-triangles four ($\alpha\beta\gamma$)-triangles

Figure 3.3: Coxeter diagrams.

without loss of generality the two γ-edges are contained in the triangle; see Figure 3.3c). This diagram has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, in contradiction with Corollary 3.12.

In case (4) the α-edges form a four-cycle as in case (1). Then each of the β-edges is incident to two ($\alpha\alpha\beta$)-triangles, so the β-edges form the diagonals of the four-cycle and the remaining star is formed by γ-edges; see Figure 3.3d). This diagram has a D_4 symmetry, in contradiction with Corollary 3.12.

In case (5) there are two possible non-isomorphic subgraphs H_α. The α-edges form a four-cycle, as in case (1), or they form a "fork", i.e., a tree with the degree sequence $(3, 2, 1, 1, 1)$. First we deal with the case they form a four-cycle; see Figure 3.3e). In order to create four ($\alpha\beta\gamma$)-triangles, the vertices of the four-cycle must be joined to the remaining vertex by two β-edges and two γ-edges, in an alternating way. This diagram has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, in contradiction with Corollary 3.12. In the "fork"-case, the β-edges are uniquely determined, since the α-edges form exactly four induced paths of length 2. The remaining two edges are γ-edges; see Figure 3.3f). This diagram has a $\mathbb{Z}_2$-symmetry; a contradiction with Corollary 3.12.

We have finished the proof of the following statement.

Proposition 3.15. *For $k \neq m^2$, every k-reptile four-dimensional simplex contains exactly one indivisible edge-angle.* □

3.4.2 Basic facts and observations from spherical geometry

Recall that all spherical triangles are regarded as subsets of the 2-dimensional unit sphere. In this subsection we assume that $\mathcal{T}$ is a spherical triangle with angles $\alpha \leq \beta \leq \gamma < \pi$ and corresponding opposite edges a, b, c. The lengths of the edges are measured in radians and again denoted by a, b, c, respectively.

The following lemma lists a few standard facts about spherical triangles (see, for example, [Zwi02]). The proof of part (a) can be found in [Pak10, Chapter 41].

Lemma 3.16. *For a spherical triangle $\mathcal{T}$, we have*

(a) $\alpha + \beta + \gamma > \pi$*, and* $\alpha + \beta + \gamma - \pi$ *is equal to the spherical area* $\Delta(\mathcal{T})$ *of* $\mathcal{T}$*.*

(b) $\beta + \gamma < \pi + \alpha$*; equivalently,* $\Delta(\mathcal{T}) < 2\alpha$ *(spherical triangle inequality).*

(c) $\cos\gamma = -\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos c$ *(spherical law of cosines for angles).*

(d) *If* $\alpha < \beta < \gamma$*, then* $a < b < c$*. If* $\alpha = \beta < \gamma$*, then* $a = b < c$*. If* $\alpha < \beta = \gamma$*, then* $a < b = c$*.*

(e) $a < b+c$*,* $b < a+c$ *and* $c < a+b$ *(triangle inequality for the spherical distance).*

(f) $a, b, c < \pi$*.*

The quantity $\alpha + \beta + \gamma - \pi$ is also called the *spherical excess* of $\mathcal{T}$.

A *spherical lune* $\mathcal{L}$ with angle $\varphi < \pi$, which we shortly call the φ*-lune*, is a slice of the sphere bounded by two half great circles whose supporting planes have dihedral angle φ. In other words, $\mathcal{L}$ is a spherical 2-gon whose vertices are two antipodal points and both inner angles are equal to φ. The spherical area of $\mathcal{L}$ is 2φ. Note that $\mathcal{L}$ contains every spherical triangle with angle φ; this implies the spherical triangle inequality (Lemma 3.16(b)).

Consider a tiling of a lune $\mathcal{L}$ by spherical triangles. A tile $\mathcal{T}$ is called a *corner tile* if $\mathcal{T}$ shares a vertex v with $\mathcal{L}$. A tile $\mathcal{T}$ is called *corner-filling* if the complement of $\mathcal{T}$ in $\mathcal{L}$ is a spherical triangle. In particular, $\mathcal{T}$ shares a vertex v with $\mathcal{L}$ and the two other vertices of $\mathcal{T}$ are internal points of the edges of $\mathcal{L}$. See Figure 3.4.

Observation 3.17. *Let* φ *be the minimum angle of a spherical triangle* $\mathcal{T}$ *and let* f *be the edge opposite to* φ*. Then in every tiling of the* φ*-lune by the copies of* $\mathcal{T}$ *there are two corner-filling tiles. Moreover, each of the corner-filling tiles neighbors with exactly one other tile, sharing the edge* f*.*

Proof. Since φ is the minimum angle of $\mathcal{T}$, every corner tile must be corner-filling. By Lemma 3.16(f), a corner-filling tile contains only one vertex of the lune, hence there are at least two such tiles. By Lemma 3.16(d), f is the shortest edge of $\mathcal{T}$. The rest of the observation follows. □

3.4.3 One indivisible edge-angle

By Lemma 3.10 and Proposition 3.15, the simplex S has only one indivisible edge-angle $\mathcal{T}_0$. This means that all the remaining edge-angles of S can by tiled with $\mathcal{T}_0$. In particular, the spherical area of every spherical triangle representing an edge-angle of S is an integer multiple of the spherical area of $\mathcal{T}_0$. Let T_0 be the triangle in $c(S)$ corresponding to the spherical triangle $\mathcal{T}_0$.

We say that a spherical triangle $\mathcal{T}$ has *type* $(\varphi\psi\chi)$ if its internal angles are φ, ψ and χ, in this case we write $\mathcal{T} = (\varphi\psi\chi)$. We sometimes write the type of $\mathcal{T}$ as (φ, ψ, χ), to avoid confusion when substituting linear combinations of angles. We say that $\mathcal{T}$ has type $(\varphi**)$ or $(\varphi\psi*)$ if it has type $(\varphi\psi\chi)$ for some angles ψ, χ, which may also be equal to φ or to each other. Note that a spherical triangle $\mathcal{T} = (\varphi\psi\chi)$ corresponds to a triangle $T = (\varphi\psi\chi)$ in $c(S)$. Since we will investigate Coxeter diagrams of S, we will, in order to simplify the notation, label the vertices of $c(S)$ by u, v, w, x, y instead of $F_1, \ldots, F_5$.

A Coxeter diagram is *rich* (or *T-rich*) if it has a copy of a same triangle T in four different orbits of triangles under the action of its symmetry group. Lemma 3.10 and Debrunner's lemma imply the following important fact.

Fact 3.18. *The Coxeter diagram of S is T_0-rich.*

A spherical triangle $\mathcal{T}$ is *realizable* if $\mathcal{T}$ can be tiled with $\mathcal{T}_0$. A triangle T is *realizable* if its corresponding spherical triangle $\mathcal{T}$ is realizable.

The strategy of the proof is the following.

- Find all possible types of $\mathcal{T}_0$.
- For every such triangle $\mathcal{T}_0$, let α be the minimal angle in $\mathcal{T}_0$ (and β the second minimal angle, if applicable). Investigate which spherical triangles of type $(\alpha**)$ (or $(\beta**)$, if needed) are realizable.
- Find all T_0-rich Coxeter diagrams whose all $(\alpha**)$-triangles (and $(\beta**)$-triangles) are realizable.
- Verify that such diagrams do not satisfy Fiedler's theorem.

We start with a simple observation about the Coxeter diagram of S.

Observation 3.19. *The Coxeter diagram of S has at least two different types of triangles.*

Proof. Suppose for contrary that all triangles in $c(S)$ are of the same type $T = (\varphi_1\varphi_2\varphi_3)$. By double-counting, the numbers of occurrences of the edge types in $c(S)$ are in the same ratio as in T. Since the numbers 10 and 3 are relatively prime, it follows that $\varphi_1 = \varphi_2 = \varphi_3$ and thus all dihedral angles in S are equal. But then S is the regular simplex, which contradicts Lemma 3.10. □

Conditions on dihedral angles

Here we prove several facts about the dihedral angles of S, which we use further to restrict the set of possible types of the indivisible triangle $\mathcal{T}_0$.

Lemma 3.20. *Let $\varphi_1, \varphi_2, \varphi_3$ be the angles of $\mathcal{T}_0$. Then for every $i \in \{1, 2, 3\}$, the spherical lune with angle φ_i can be tiled with $\mathcal{T}_0$.*

Proof. Fix $i \in \{1, 2, 3\}$. If S is a k-reptile simplex for some $k > 1$, then by induction, S is k^n-reptile for every $n \geq 1$. In particular, there is a tiling of S with simplices similar to S where some of the tiles, S', has an edge e that is contained in the interior of a 2-face of S with dihedral angle φ_i. Select an interior point x_i of e that misses all vertices of all the tiles. Let h_i be the hyperplane orthogonal to e and containing x_i. In a small neighborhood of x_i in h_i, the tiles with x_i on their boundary induce a tiling of a wedge with angle φ_i by triangular cones originating in x_i and possibly by wedges with angles $\psi_j < \varphi_i$, where ψ_j is an internal angle of some realizable triangle $\mathcal{T}$.

This is analogous to the situation in a tiling of a three-dimensional simplex in the neighborhood of an internal point x of an edge such that x is also a vertex of some tile. In the intersection of h_i with a small sphere centered in x_i, we thus obtain a

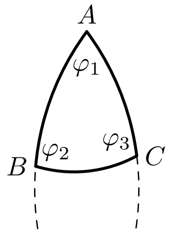


Figure 3.4: A corner-filling tile in the φ_1-lune.

tiling of the spherical lune with angle φ_i by spherical triangles corresponding to edge-angles of the tiles and possibly by spherical lunes with angles $\varphi_j < \varphi_i$ corresponding to dihedral angles of the tiles. Since $\mathcal{T}$ can be tiled with $\mathcal{T}_0$ (recall that $\mathcal{T}_0$ is the only indivisible edge-angle), it follows that ψ_j is a nontrivial linear combination of φ_k, φ_ℓ, where $0 \le k, \ell < i$. Hence we can assume that any wedge with angle ψ_j can be, in fact, tiled by wedges with angles φ_j, where $j < i$. Note that the described tiling of a wedge with angle φ_i induces a tiling of the corresponding spherical lune with angle φ_i. Observe that since x_i is an internal point of an edge of at least one tile, the tiling contains at least one spherical triangle.

Suppose that $\varphi_1 \le \varphi_2 \le \varphi_3$. Then for $i = 1$, the tiling consists solely of realizable spherical triangles. For $i > 1$, the tiling consists of realizable spherical triangles and possibly φ_j-lunes with $j < i$. The lemma follows by induction on i. □

The following statement is a stronger variant of the Bricard's condition for equidecomposable polyhedra [AZ10, Ben07], [Pak10, Chapter 15].

Lemma 3.21. *Let $\{\varphi_1, \varphi_2, \varphi_3\}$ be the set of angles of $\mathcal{T}_0$. Then for every $i \in \{1, 2, 3\}$, there exist nonnegative integers $m_1 = m_1(i), m_2 = m_2(i), m_3 = m_3(i)$ such that $m_i > 0$ and $m_1\varphi_1 + m_2\varphi_2 + m_3\varphi_3 = \pi$.*

Proof. Let $\varphi_1 \le \varphi_2 \le \varphi_3$. By Lemma 3.20, the φ_1-lune $\mathcal{L}_1$ is tiled with $\mathcal{T}_0$. By Observation 3.17, there is a corner-filling tile $\mathcal{T}_0^1$ whose vertices with inner angles φ_2 and φ_3 are internal points of the edges of $\mathcal{L}_1$; see Figure 3.4. In a small neighborhood of each of these two points we observe a tiling of the straight angle by the angles of $\mathcal{T}_0$, including φ_2 or φ_3, respectively. This shows the lemma for $i = 2$ and $i = 3$.

To show the lemma for $i = 1$, we distinguish two cases. If φ_1 divides φ_2, the claim follows from the case $i = 2$. Otherwise, we use Lemma 3.20 again, now for the φ_2-lune $\mathcal{L}_2$. The argument is analogous to the previous case since in the tiling of $\mathcal{L}_2$ with $\mathcal{T}_0$ each corner tile is corner-filling. □

For the rest of the chapter, let us assume that α is the minimum angle of the indivisible edge-angle $\mathcal{T}_0$.

Corollary 3.22. *We have $\alpha < \pi/2$.*

Proof. Lemma 3.21 implies that $\alpha \le \pi/2$. By Lemma 3.20, the α-lune is tiled with at least two copies of $\mathcal{T}_0$. Now suppose that $\alpha = \pi/2$. Since $\Delta(\mathcal{T}_0) \ge \pi/2$ and the spherical area of the $\pi/2$-lune is π, the tiling of the $\pi/2$-lune consists of precisely two tiles $\mathcal{T}_0$ and $\Delta(\mathcal{T}_0) = \pi/2$. This means that $\mathcal{T}_0 = (\pi/2, \pi/2, \pi/2)$. The angles of every spherical triangle tiled with $\mathcal{T}_0$ must be integer multiples of $\pi/2$, thus $\mathcal{T}_0$ is the only realizable triangle. This contradicts Observation 3.19. □

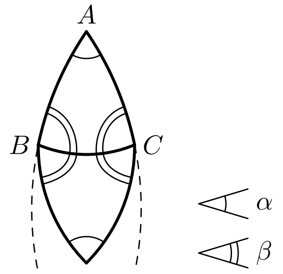


Figure 3.5: A corner-filling $\alpha\beta\beta$-tile and its adjacent tile in the α-lune.

Corollary 3.23. *We cannot have* $\mathcal{T}_0 = (\alpha\alpha\alpha)$.

Proof. If $\mathcal{T}_0 = (\alpha\alpha\alpha)$, then by Lemma 3.21, $\alpha = \pi/n$ for some positive integer n. But we have $\alpha > \pi/3$ by Lemma 3.16(a) and $\alpha < \pi/2$ by Corollary 3.22; a contradiction. □

We are left with three main cases for the type of the indivisible triangle $\mathcal{T}_0$, according to the symmetries and relative sizes of its angles.

(A) Exactly two angles in $\mathcal{T}_0$ are equal, but not to α. We write $\mathcal{T}_0 = (\alpha\beta\beta)$.

(B) Exactly two angles in $\mathcal{T}_0$ are equal to α. We write $\mathcal{T}_0 = (\alpha\alpha\beta)$.

(C) All three angles in $\mathcal{T}_0$ are different. We write $\mathcal{T}_0 = (\alpha\beta\gamma)$.

For the rest of the chapter we assume that $\alpha < \beta < \gamma$.

Case (A): $\mathcal{T}_0 = (\alpha\beta\beta)$**.**

By Lemma 3.20, the α-lune $\mathcal{L}_\alpha$ can be tiled with $\mathcal{T}_0$. By Observation 3.17, there is a corner-filling tile $\mathcal{T}_0^1$ sharing its shortest edge with another tile $\mathcal{T}_0^2$. See Figure 3.5. In the neighborhood of either common vertex of $\mathcal{T}_0^1$ and $\mathcal{T}_0^2$, we see the straight angle tiled with two angles β and possibly other angles α or β. Since $\alpha + 2\beta > \pi$ by Lemma 3.16(a), two angles β already tile the straight angle and hence $\beta = \pi/2$. It follows that two copies of $\mathcal{T}_0$ tile the whole α-lune and thus $\mathcal{T}_0$ is the only realizable spherical triangle of type $(\alpha **)$. (We recall that $(\alpha **)$ stands for $(\alpha\varphi\psi)$, where φ, ψ may also be equal to α or to each other.)

This implies that the Coxeter diagram of S has two vertex-disjoint α-edges, since it is $(\alpha\beta\beta)$-rich and has no $(\alpha\alpha*)$-triangle. Every remaining edge is adjacent to at least one α-edge, hence it is a β-edge. The resulting diagram has only two orbits of $(\alpha\beta\beta)$-triangles and so it is not $(\alpha\beta\beta)$-rich, a contradiction.

Case (B): $\mathcal{T}_0 = (\alpha\alpha\beta)$**.**

We start with an observation about $(\alpha\alpha\beta)$-rich Coxeter diagrams.

Lemma 3.24. *There is at least one realizable spherical* $(\alpha **)$*-triangle different from* $\mathcal{T}_0$.

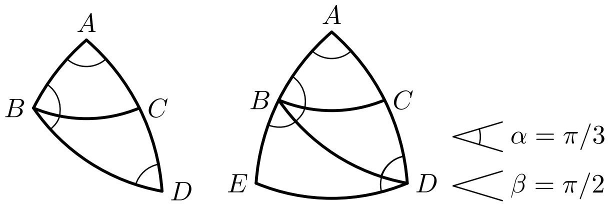


Figure 3.6: The $(\alpha\alpha 2\alpha)$-triangle and the $(\alpha, 2\alpha, \pi/2)$-triangle composed of two and three $(\alpha\alpha\beta)$-tiles, respectively.

Proof. If $\mathcal{T}_0 = (\alpha\alpha\beta)$ is the only realizable $(\alpha**)$ triangle, then the α-edges form a spanning complete bipartite subgraph in the Coxeter diagram of S and all the remaining edges are β-edges. The $(\alpha\alpha\beta)$-triangles then form at most two orbits, a contradiction. □

By Lemma 3.21, there exist integers $m_1 \geq 0$ and $m_2 \geq 1$ such that $m_1\alpha + m_2\beta = \pi$. We distinguish two cases.

(1) $m_1 \neq 0$. Since $2\alpha + \beta > \pi$ by Lemma 3.16(a), we have $m_1 = m_2 = 1$ and thus $\alpha + \beta = \pi$.

(2) $m_1 = 0$. Then $\beta = \pi/m_2$. Since $3\beta > \pi$ and $\beta < \pi$, we have $m_2 = 2$ and so $\beta = \pi/2$. Now the inequality $2\alpha + \beta > \pi$ implies that $\alpha > \pi/4$. By Lemma 3.21, there exist integers $m'_1 \geq 1$ and $m'_2 \geq 0$ such that $m'_1\alpha + m'_2\pi/2 = \pi$. Since $m'_2 > 0$ leads to contradiction, we have $\alpha = \pi/m'_1$. The only solution satisfying $\pi/4 < \alpha < \pi/2$ is $\alpha = \pi/3$.

Case (1): $\alpha + \beta = \pi$**.** Let $\mathcal{T} = (\alpha\varphi\psi)$ be the realizable spherical $(\alpha**)$-triangle different from $\mathcal{T}_0$ (with some of the angles possibly equal), guaranteed by Lemma 3.24. The spherical area of $\mathcal{T}$ satisfies $\Delta(\mathcal{T}) \geq 2\Delta(\mathcal{T}_0) = 2 \cdot (2\alpha + \beta - \pi) = 2\alpha$. But this contradicts the spherical triangle inequality $\Delta(\mathcal{T}) < 2\alpha$ (Lemma 3.16(b)).

Case (2): $\alpha = \pi/3, \beta = \pi/2$**.** The only nontrivial nonnegative integer combinations of α and β that sum up to π are 3α and 2β. This implies that in every tiling of a spherical polygon $\mathcal{P}$ by $\mathcal{T}_0$, for every internal point x of an edge of $\mathcal{P}$, all incident tiles have the same angle at x. This somewhat restricts the set of possible tilings. Further restriction is obtained using the area argument.

The spherical area of $\mathcal{T}_0$ is $\beta + 2\alpha - \pi = \pi/6$. The α-lune, which contains every $(\alpha**)$-triangle, has spherical area $2\alpha = 2\pi/3$. It follows that every $(\alpha**)$-triangle is composed of at most three tiles.

When constructing a tiling of an $(\alpha**)$-triangle, we always start with a corner-filling tile $\mathcal{T}_0^1$ of the α-lune and then try to place additional tiles. There is only one way of attaching a second tile to $\mathcal{T}_0^1$, yielding the triangle of type $(\alpha, \alpha, 2\alpha)$; see Figure 3.6, left. Similarly, there is a unique way of attaching the third tile, which yields the triangle of type $(\alpha, 2\alpha, \pi/2)$; see Figure 3.6, right. These are the only realizable $(\alpha**)$-triangles other than $\mathcal{T}_0$.

Realizable $(\beta**)$-triangles can be composed from at most five tiles, as their spherical area is smaller than $2\beta = \pi$. To construct a $(\beta**)$-triangle, we start with a corner tile $\mathcal{T}_0^1$ in the β-lune. Since α does not divide β, the corner tile is corner-filling. By

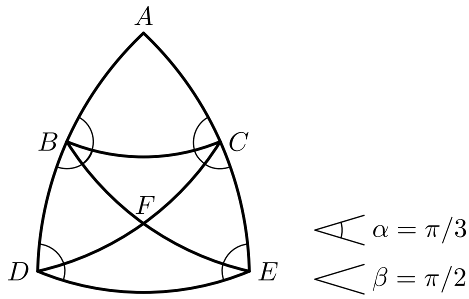


Figure 3.7: The $(\beta 2\alpha 2\alpha)$-triangle composed of five $(\alpha\alpha\beta)$-tiles with $\alpha = \pi/3, \beta = \pi/2$.

Lemma 3.16(e) and by the symmetry of $\mathcal{T}_0$, the longest edge of $\mathcal{T}_0$ cannot be tiled with the shorter edges. This means that there is just one possible way of attaching another tile, $\mathcal{T}_0^2$, to $\mathcal{T}_0^1$; see Figure 3.7, where $\mathcal{T}_0^1$ has vertices ABC and $\mathcal{T}_0^2$ has vertices BCF. The two tiles do not form a triangle, yet. Hence, there is at least one more tile $\mathcal{T}_0^3$ adjacent to, say, B. The orientation of $\mathcal{T}_0^3$ where $\mathcal{T}_0^3$ shares the edge BF with $\mathcal{T}_0^2$ gives an $(\alpha, 2\alpha, \pi/2)$-triangle obtained earlier. The other orientation of $\mathcal{T}_0^3$, where the longest edge of $\mathcal{T}_0^3$ partially coincides with the edge BF, forces a fourth tile sharing the edge CF with $\mathcal{T}_0^2$, forming an $(\alpha, 2\alpha, \pi/2)$-triangle ABE with $\mathcal{T}_0^1$ and $\mathcal{T}_0^2$. The remaining uncovered part of the edge BE is shorter than all edges of $\mathcal{T}_0$, thus such a tiling cannot be completed to a $(\beta * *)$-triangle.

To extend the $(\alpha, 2\alpha, \pi/2)$-triangle ACD, at least two more tiles are needed. There is precisely one way of attaching two more tiles, giving a $(\beta, 2\alpha, 2\alpha)$-triangle composed of five pieces; see Figure 3.7. Therefore, the only $(\beta * *)$-triangles are $(\alpha\alpha\beta)$, $(\alpha, 2\alpha, \beta)$ and $(\beta, 2\alpha, 2\alpha)$. In particular, there is no $(\beta\beta *)$-triangle.

This implies that the Coxeter diagram of S has exactly two vertex-disjoint β-edges. Let uv and xy be the two β-edges and let w be the fifth vertex of $c(S)$. If both triangles uvw and xyw are $(\alpha\alpha\beta)$-triangles, then all edges incident with w are α-edges. No other edge can be an α-edge, since the $(\alpha\alpha\alpha)$-triangle is not realizable. In other words, the triangles uvw and xyw are the only $(\alpha\alpha\beta)$-triangles; a contradiction. Hence at least three of the triangles induced by the vertices u, v, x, y are $(\alpha\alpha\beta)$-triangles. In fact, then all four of the triangles are $(\alpha\alpha\beta)$-triangles since the edges ux, uy, vx, vy must be α-edges. Every triangle containing one of these four α-edges and the vertex w must be of type $(\alpha, \alpha, 2\alpha)$. Therefore, without loss of generality, the edges uw and vw are α-edges, and the edges xw and yw are 2α-edges. The diagram looks like the one in Figure 3.3c). But in such a diagram, there are only three orbits of $(\alpha\alpha\beta)$-triangles; a contradiction.

Case (C): $\mathcal{T}_0 = (\alpha\beta\gamma)$.

First we obtain some more information about the Coxeter diagram of S.

Lemma 3.25. *There is at least one realizable spherical $(\alpha * *)$-triangle different from $\mathcal{T}_0$. The same is true for triangles of type $(\beta * *)$ and $(\gamma * *)$.*

Proof. Assume for contradiction that $\mathcal{T}_0$ is the only realizable spherical triangle of type $(\alpha * *)$. It follows that there are exactly two vertex-disjoint α-edges in $c(S)$. Since every other edge in $c(S)$ is adjacent to an α-edge, all edges in $c(S)$ are β-edges or

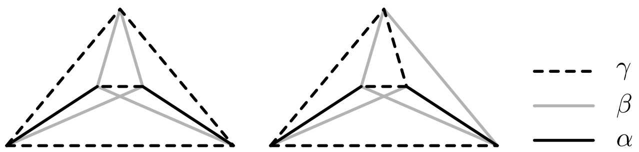


Figure 3.8: Two possible Coxeter diagrams if $\mathcal{T}_0$ is the only triangle of type $(\alpha**)$.

γ-edges, and $c(S)$ is isomorphic to one of the two diagrams in Figure 3.8. In both diagrams, the six $(\alpha\beta\gamma)$-triangles form only three orbits; a contradiction. □

Recall that P_k denotes a path with k vertices and $P_k + P_l$ denotes a disjoint union of paths.

Lemma 3.26. *The Coxeter diagram of S has two or three α-edges and they form a subgraph isomorphic to $P_2 + P_2$ or $P_2 + P_3$.*

Proof. Let $H = c(S)$ be the Coxeter diagram of S. Let $V(H) = \{u, v, w, x, y\}$ and let $H_\alpha = (V(G), E_\alpha)$ be the subgraph of H formed by the α-edges. Since H is $(\alpha\beta\gamma)$-rich, it has at least four $(\alpha\beta\gamma)$-triangles, and hence it has at least two α-edges, at least two β-edges and at least two γ-edges.

Suppose that $|E_\alpha| = 2$. If the two α-edges are adjacent, say, $E_\alpha = \{uv, uw\}$, then all the triangles uvx, uvy, uwx, uwy are of type $(\alpha\beta\gamma)$. In particular, the edges vx and wx are of the same type, either β or γ, and the edges vy and wy are of the same type as well. Therefore, H has a symmetry switching v with w, and so there are at most two orbits of $(\alpha\beta\gamma)$-triangles; a contradiction. It follows that H_α is a matching.

Suppose that $|E_\alpha| = 3$. If H_α is isomorphic to the star $K_{1,3}$, say, $E_\alpha = \{xu, xv, xw\}$, then every $(\alpha\beta\gamma)$-triangle must contain the vertex y, so there can be at most three such triangles. If H_α is isomorphic to the path P_4, say, $E_\alpha = \{xu, uv, vw\}$, then the edges xv and uw cannot have type β or γ, since the spherical triangles of type $(\alpha\alpha\beta)$ and $(\alpha\alpha\gamma)$ have smaller area than $\mathcal{T}_0$ and so they are not realizable. This again implies that every $(\alpha\beta\gamma)$-triangle must contain the vertex y and so there are at most three of them. Also, H_α cannot form a triangle, since the spherical triangle of type $(\alpha\alpha\alpha)$ is not realizable. This leaves only the last option: H_α forms a subgraph isomorphic to $P_2 + P_3$.

Suppose that $|E_\alpha| \geq 4$. If H_α contains a star $K_{1,4}$, then no other edge can be of type β or γ. If H_α contains a "fork", say, $E_\alpha \supseteq \{uv, vw, wx, wy\}$, then only two edges, ux and uy, can be of type β or γ. If H_α contains a path P_5, say, $E_\alpha \supseteq \{uv, vw, wx, xy\}$, then only three edges, ux, uy and vy, can be of type β or γ. Since H_α cannot contain triangles, the last possibility is that H_α is isomorphic to the 4-cycle, say, $E_\alpha = \{uv, vx, xy, yu\}$. All edges of type β or γ must be incident with w, hence uw and xw are of the same type, and also vw and yw are of the same type. Regardless of the type of the diagonals ux and vy, this diagram has a symmetry group generated by the transpositions (u, x) and (v, y), and so the $(\alpha\beta\gamma)$-triangles form just one orbit; a contradiction. □

In the following lemma we obtain some partial information about the angles of $\mathcal{T}_0$ and identify two basic cases.

Lemma 3.27. *If $\mathcal{T}_0 = (\alpha\beta\gamma)$ then $\gamma = \pi/2$. Furthermore,*

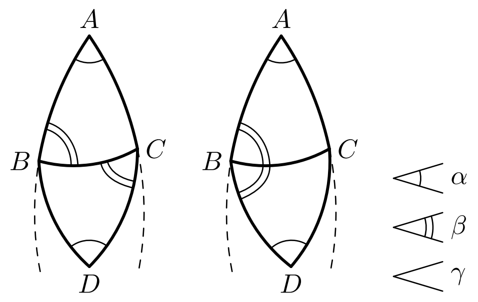


Figure 3.9: Two possibilities for the first two tiles in the tiling of $\mathcal{L}_\alpha$ by $\mathcal{T}_0$.

a) $\alpha + 2\beta = \pi$*, or*

b) $\beta = \pi/3$ *and* $\alpha > \pi/6$*.*

Proof. By Lemma 3.25, the spherical area of the α-lune $\mathcal{L}_\alpha$ is greater than $2\Delta(\mathcal{T}_0)$. By Lemma 3.20, there is a tiling of the lune $\mathcal{L}_\alpha$ by at least three copies of $\mathcal{T}_0$. Let $\mathcal{T}_0^1$ be a corner-filling tile with vertices A, B, C incident with angles α, β, γ, respectively. In particular, A is a vertex of $\mathcal{L}_\alpha$. By Observation 3.17, $\mathcal{T}_0^1$ is adjacent to a tile $\mathcal{T}_0^2$ with vertices B, C, D, which can be placed in two possible orientations; see Figure 3.9. If $\mathcal{T}_0^2$ has the same orientation as $\mathcal{T}_0^1$, the quadrilateral $ABDC$ is a spherical parallelogram with angles $\beta + \gamma$ at vertices B and C. By Lemma 3.25, two copies of $\mathcal{T}_0$ cannot tile $\mathcal{L}_\alpha$ and so $\beta + \gamma < \pi$. Since $\alpha + \beta + \gamma > \pi$ by Lemma 3.16(a), the parallelogram $ABDC$ cannot be completed to a tiling of $\mathcal{L}_\alpha$. Therefore $\mathcal{T}_0^2$ and $\mathcal{T}_0^1$ have opposite orientations.

Since $\alpha + 2\gamma > \alpha + \beta + \gamma > \pi$, no other tile can be incident to C and so $\gamma = \pi/2$. That is, the two tiles $\mathcal{T}_0^1$ and $\mathcal{T}_0^2$ form a triangle ABD. The angles of the tiles incident to B include two angles β, and together they sum up to π. No tile can have angle γ at B since $2\beta + \gamma > \pi$. Therefore, there exist non-negative integers n_1 and $n_2 \geq 2$ such that $n_1\alpha + n_2\beta = \pi$. Since $\gamma = \pi/2$, we have $\alpha + \beta > \pi/2$, implying $n_1 \leq 1$ and $\beta > \pi/4$. There are only two cases:

a) $n_1 = 1$: then $n_2 = 2$ and thus $\alpha + 2\beta = \pi$.

b) $n_1 = 0$: then $\beta = \pi/3$ and consequently $\alpha > \pi/6$.

This concludes the proof. □

Now we deal separately with the two cases from Lemma 3.27.

Case a) $\alpha + 2\beta = \pi$**.** By Observation 3.17 and by the fact that $\beta + \gamma < \pi < \alpha + \beta + \gamma$, the tiling of every realizable $(\alpha**)$-triangle other than $\mathcal{T}_0$ contains two tiles ABC and BCD with opposite orientations as in Figure 3.10, forming a triangle of type $(\alpha, \alpha, 2\beta)$. By the triangle inequality (Lemma 3.16(e)), the edge BD cannot be subdivided by the edge of $\mathcal{T}_0$ opposite to β, thus there is just one possible way of placing a third tile: the triangle BDE in Figure 3.10. These three tiles form a triangle of type $(\alpha, \alpha + \beta, \gamma)$. The fourth tile would fill the whole α-lune, therefore the only realizable $(\alpha**)$-triangles are $(\alpha\beta\gamma)$, $(\alpha, \alpha, 2\beta)$ and $(\alpha, \alpha + \beta, \gamma)$.

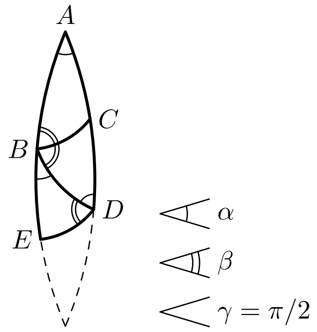


Figure 3.10: Tiling of spherical triangles for $\gamma = \pi/2$ and $\alpha + 2\beta = \pi$.

Lemma 3.28. *Every $(\alpha\beta\gamma)$-rich Coxeter diagram with five vertices where all $(\alpha**)$-triangles are of type $(\alpha\beta\gamma)$, $(\alpha, \alpha, 2\beta)$ or $(\alpha, \alpha+\beta, \gamma)$, is isomorphic to one of the five diagrams in Figure 3.11.*

Proof. Let H be a Coxeter diagram satisfying the assumptions of the lemma. Let $V(H) = \{u, v, w, x, y\}$ and let E_α be the set of α-edges. By Lemma 3.26, we distinguish two cases, up to isomorphism.

1) $E_\alpha = \{ux, vy\}$. In this case, all triangles containing an α-edge are of type $(\alpha\beta\gamma)$ or $(\alpha, \alpha+\beta, \gamma)$. In particular, every such triangle contains exactly one γ-edge. By symmetry, we may assume that xw and yw are γ-edges. The other two γ-edges form a matching on vertices u, v, x, y; there are two possibilities, $\{uv, xy\}$ and $\{uy, vx\}$. The remaining four edges are of type β or $\alpha+\beta$. If at least one of the two edges uw and vw is of type $\alpha+\beta$, the remaining edges must be β-edges so that there are at least four $(\alpha\beta\gamma)$-triangles. However, if both edges uw and vw are of type $\alpha+\beta$, then due to the symmetry Φ exchanging simultaneously u with v and x with y, the $(\alpha\beta\gamma)$-triangles form only two orbits. Thus only one edge, say, vw, is of type $\alpha+\beta$ and we have the diagram in Figure 3.11(a) or (b).

In the other case both edges uw and vw are of type β. If both remaining edges are of type β, then due to the symmetry Φ the $(\alpha\beta\gamma)$-triangles form only three orbits. Therefore, exactly one edge is of type $\alpha+\beta$. However, if uy and vx were chosen as γ-edges, the diagram has again Φ as its symmetry and the $(\alpha\beta\gamma)$-triangles form only two orbits. So in this case we obtain only one diagram; see Figure 3.11(c).

2) $E_\alpha = \{uw, vw, xy\}$. In this case the edge uv must be of type 2β and it is the only edge of this type. All seven $(\alpha**)$-triangles other than uvw have exactly one γ-edge. Due to symmetry, we may assume that xw is a γ-edge. Then yu and yv must be γ-edges as well. The remaining three edges, yw, xu and xv, are of type β or $\alpha+\beta$. If all these three edges are of type β, the diagram has a symmetry Ψ exchanging u and v, but the $(\alpha\beta\gamma)$-triangles still form four orbits; see Figure 3.11(d). Otherwise, exactly one of the three edges yw, xu, xv is of type $\alpha+\beta$. If yw is of type $\alpha+\beta$, then due to the symmetry Ψ, the $(\alpha\beta\gamma)$-triangles form only two orbits. The other two cases give isomorphic diagrams; see Figure 3.11(e). □

We immediately notice that the diagram in Figure 3.11(e) cannot be a Coxeter diagram of S, since xuv is a $(\beta, \alpha+\beta, 2\beta)$-triangle but there is no spherical triangle of type $(\beta, \alpha+\beta, 2\beta)$ by Lemma 3.16(b).

We are left with diagrams in Figure 3.11(a)–(d). Since investigating realizable

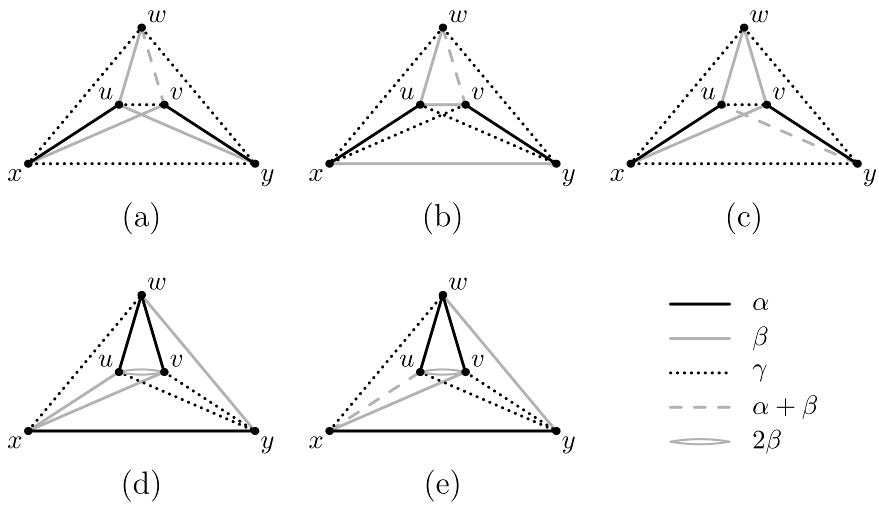


Figure 3.11: Coxeter diagrams for the case $\alpha + 2\beta = \pi$.

$(\beta * *)$-triangles does not seem to help much, we proceed to the next step and use Fiedler's theorem (Theorem 3.5).

Recall that the matrix A associated to a simplex S has -1's on the diagonal and $a_{ij} = \cos\beta_{ij}$ for $i \neq j$, where β_{ij} is the dihedral angle between facets F_i and F_j. Note that the matrix A is completely determined by $c(S)$: the angle β_{ij} is the label of the edge F_iF_j in $c(S)$.

Let $t := \cos\beta$ and $s := \cos\alpha$. Since $\alpha+2\beta = \pi$ and $\alpha < \beta$, it follows that $s = 1-2t^2$ and $t \in (0, 1/2)$. Moreover, $\cos 2\beta = 2t^2 - 1$ and $\cos(\alpha + \beta) = \cos(\pi - \beta) = -t$.

The following matrices $A_1, \ldots, A_4$ are associated to simplices represented by diagrams in Figure 3.11(a)–(d), respectively. The rows and columns of $A_1, \ldots, A_4$ are indexed by u, v, w, x, y (in this order), where the vertices u, v, w, x, y of $c(S)$ represent facets $F_1, \ldots, F_5$ in S.

$$A_1 = \begin{pmatrix} -1 & 0 & t & 1-2t^2 & t \\ 0 & -1 & -t & t & 1-2t^2 \\ t & -t & -1 & 0 & 0 \\ 1-2t^2 & t & 0 & -1 & 0 \\ t & 1-2t^2 & 0 & 0 & -1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -1 & t & t & 1-2t^2 & 0 \\ t & -1 & -t & 0 & 1-2t^2 \\ t & -t & -1 & 0 & 0 \\ 1-2t^2 & 0 & 0 & -1 & t \\ 0 & 1-2t^2 & 0 & t & -1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} -1 & 0 & t & 1-2t^2 & -t \\ 0 & -1 & t & t & 1-2t^2 \\ t & t & -1 & 0 & 0 \\ 1-2t^2 & t & 0 & -1 & 0 \\ -t & 1-2t^2 & 0 & 0 & -1 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} -1 & 2t^2-1 & 1-2t^2 & t & 0 \\ 2t^2-1 & -1 & 1-2t^2 & t & 0 \\ 1-2t^2 & 1-2t^2 & -1 & 0 & t \\ t & t & 0 & -1 & 1-2t^2 \\ 0 & 0 & t & 1-2t^2 & -1 \end{pmatrix}$$

Considering t as a variable, the determinants of matrices $A_1, \ldots, A_4$ are polynomials in t. Let $\Gamma(A_i)$ be the set of roots of the determinant of A_i. Rounding the roots to two decimal places, we have:

A	$\det(A)$	$\Gamma(A)$
A_1	$-t^2(2t-1)(2t^2-t-2)(4t^3+4t^2-t-2)$	$\{0, 0.5, 2.56, -1.56, -0.78, 1.28, 0.63\}$
A_2	$-t^2(2t-1)(2t^2+t-2)(4t^3+2t^2-3t-2)$	$\{0, 0.5, 2.56, -1.56, -1.28, 0.78, 0.92\}$
A_3	$-t^4(2t-1)(2t+1)(4t^2-3)$	$\{0, \pm 0.5, \pm 0.87\}$
A_4	$-8t^4(2t^2-1)(4t^4-7t^2+2)$	$\{0, \pm 0.71, \pm 1.18, \pm 0.60\}$

By Fiedler's theorem, the matrix associated to a simplex is singular. Therefore, the determinant of A_i must have a root in the interval $(0, 1/2)$. Since no A_i satisfies this condition, we have a contradiction and the lemma follows.

Case b) $\beta = \pi/3$ **and** $\alpha > \pi/6$**.**

Lemma 3.29. *If $\mathcal{T}_0 = (\alpha, \pi/3, \pi/2)$ with $\alpha > \pi/6$, then $\alpha \in \{\pi/4, 2\pi/9, \pi/5\}$.*

Proof. By Lemma 3.21, there are integers $m \geq 1$ and $n \geq 0$ such that $m\alpha + n\pi/3 = \pi$. Since $\pi/6 < \alpha < \pi/3$, we have $n \leq 1$. For $n = 0$ we have $\alpha = \pi/m$, thus $m = 4$ or 5, while for $n = 1$ we have $\alpha = 2\pi/(3m)$, which is in the interval $(\pi/6, \pi/3)$ only for $m = 3$. The lemma follows. □

The following simple observation will be useful for investigating all realizable $(\alpha{*}{*})$- and $(\pi/3 * *)$-triangles.

Observation 3.30. *Let $\mathcal{L}$ be a realizable φ-lune, $\mathcal{H}$ a realizable corner-filling spherical triangle whose two copies tile the lune $\mathcal{L}$ and $\mathcal{K} \subseteq \mathcal{H}$ a realizable corner-filling spherical triangle used in a tiling of $\mathcal{H}$. Then the complement of $\mathcal{K}$ in $\mathcal{L}$ is also realizable; see Figure 3.12.* □

In the following lemma we investigate all realizable $(\alpha * *)$- and $(\beta * *)$-triangles for $\mathcal{T}_0 = (\alpha, \pi/3, \pi/2)$. For better clarity we write α and β rather than their numerical values.

Lemma 3.31. *Let $\mathcal{T}_0 = (\alpha, \beta, \pi/2)$, where $\beta = \pi/3$. Depending on the value of α, the realizable $(\alpha * *)$-triangles and $(\beta * *)$-triangles other than $\mathcal{T}_0$ are the following.*

1) $\alpha = \pi/4$

- $(\alpha, \alpha, 2\beta)$, $(\alpha, \pi/2, \pi/2)$, $(\alpha, \beta, 3\alpha)$, $(\alpha, \pi/2, 2\beta)$
- $(\beta, \beta, \pi/2)$, $(\beta, \beta, 2\beta)$, $(\beta, \pi/2, 2\beta)$, $(\beta, \pi/2, 3\alpha)$

2) $\alpha = \pi/5$

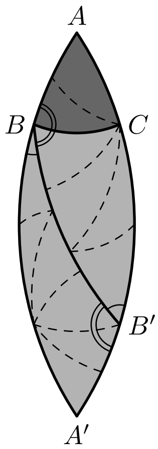


Figure 3.12: Tiling of a realizable lune $\mathcal{L}$, where $\mathcal{H} = ABB'$ and $\mathcal{K} = ABC$.

- $(\alpha, \alpha, 2\beta)$, $(\alpha, 2\alpha, \pi/2)$, $(\alpha, \beta, 3\alpha)$, $(\alpha, \beta, 2\beta)$, $(\alpha, \alpha, 4\alpha)$, $(\alpha, 2\alpha, 2\beta)$, $(\alpha, \pi/2, 3\alpha)$, $(\alpha, \beta, 4\alpha)$, $(\alpha, \pi/2, 2\beta)$
- $(\beta, \beta, 2\alpha)$, $(\beta, 2\alpha, \pi/2)$, $(\beta, 2\alpha, 3\alpha)$, $(\beta, \pi/2, 3\alpha)$, $(\beta, \beta, 4\alpha)$, $(\beta, 2\alpha, 4\alpha)$, $(\beta, 3\alpha, 2\beta)$, $(\beta, \pi/2, 4\alpha)$

3) $\alpha = 2\pi/9$

- $(\alpha, \alpha, 2\beta)$, $(\alpha, 2\alpha, \pi/2)$, $(\alpha, \beta, 2\beta)$, $(\alpha, \alpha, 4\alpha)$, $(\alpha, \pi/2, \alpha+\beta)$, $(\alpha, \beta, 2\alpha+\beta)$, $(\alpha, \pi/2, 2\beta)$
- $(\beta, \beta, 2\alpha)$, $(\beta, \alpha+\beta, 2\beta)$, $(\beta, \pi/2, 2\alpha+\beta)$.

Proof. Recall that for triangle $\mathcal{T}_0$, a, b, c denote the edges (and also their length) opposite to angles α, β, γ, respectively.

First we describe a general method how to find all realizable $(\varphi * *)$-triangles, which we lately apply to specific values of α. The rough idea is to consider all spherical triangles whose angles and edges can be obtained as a combination of angles and edges of the basic tile $\mathcal{T}_0$, respectively, and which area is a multiple of the area of $\mathcal{T}_0$.

Here we provide more details. Let $\mathcal{T} = (\tau\varphi\psi)$, $\mathcal{T} \neq \mathcal{T}_0$, be a realizable triangle. We may assume that $\mathcal{T}$ is a corner-filling tile of a τ-lune and $\varphi \leq \psi$.

Recall that $\Delta(\mathcal{T})$ denotes the spherical area of $\mathcal{T}$. Because $\mathcal{T}$ is realizable, it follows that $\Delta(\mathcal{T}) = n\Delta(\mathcal{T}_0)$, where n is a number of copies of $\mathcal{T}_0$ needed to tile $\mathcal{T}$. By Lemma 3.16(a), we have

$$n\Delta(\mathcal{T}_0) = \Delta(\mathcal{T}) = \varphi + \psi + \tau - \pi \quad (3.3)$$

$$n(\alpha + \beta + \gamma - \pi) + \pi - \tau = \varphi + \psi \quad (3.4)$$

$$n(\alpha + \beta + \gamma - \pi) + \pi - \tau = \ell_1\alpha + \ell_2\pi/3 + \ell_3\pi/2, \quad (3.5)$$

where the last equality follows from the realizability of $\mathcal{T}$. Indeed, the angles φ, ψ can be expressed as a nonnegative integral combination of angles α, β, γ, respectively.

Note that $2 \leq n < 2\tau/\Delta(\mathcal{T}_0)$, since the volume of the φ-lune is 2τ and $\mathcal{T} \neq \mathcal{T}_0$. It follows that from the solutions of the equation (3.5) we get possible forms of $\varphi + \psi$ and hence also possible pairs of φ and ψ. We note that we will consider only triples of τ, φ and ψ which correspond to a spherical triangle, that is, the values of τ, φ and ψ satisfy condition (b) in Lemma 3.16.

n	(ℓ_1, ℓ_2, ℓ_3)	(φ, ψ)	length of x	best approximation for x using a, b, c
2	(1,2,0)	$(\beta, \alpha+\beta)$	0.80848	$0.7854 \sim b < x < c \sim 0.95532$
		$(\alpha, 2\beta)$	0.95532	$x = c$
3	(0,0,2)	(γ, γ)	0.7854	$x = b$
	(0,3,0)	$(\beta, 2\beta)$	0.9154	$0.7854 \sim b < x < c \sim 0.95532$
	(2,0,1)	$(\alpha, \alpha+\gamma)$	1.14372	$0.95532 \sim c < x < 2a \sim 1.23096$
4	(1,1,1)	$(\gamma, \alpha+\beta)$	0.74947	$0.61548 \sim a < x < b \sim 0.7854$
		$(\beta, \alpha+\gamma)$	0.95532	$x = c$
		$(\alpha, \beta+\gamma)$	1.29953	$1.23096 \sim 2a < x < a+b \sim 1.40088$
5	(0,2,1)	$(\gamma, 2\beta)$	0.61548	$x = a$
		$(\beta, \beta+\gamma)$	0.88538	$0.7854 \sim b < x < c \sim 0.95532$
	(2,2,0)	$(\alpha, \alpha+2\beta)$	1.43876	$1.40088 \sim a+b < x < 2b \sim 1.5708$
		$(\alpha+\beta, \alpha+\beta)$	0.59228	$0.0 \sim 0 < x < a \sim 0.61548$

Table 3.2: $(\alpha**)$-triangles for $\alpha = \pi/4$, $a \sim 0.61548, b \sim 0.78540, c \sim 0.95532$.

Using the spherical law of cosines (Lemma 3.16(c)), we compute the length of the edge x opposite to the angle τ and check, if it can be obtained as a nonnegative combination of a, b, c. If not, $\mathcal{T} = (\tau\varphi\psi)$ is clearly not realizable.

Since the calculations are tedious and it is relatively easy to make a mistake or forget some case, we have decided to write a computer program to go through all the possibilities. The computer program was written in Sage 5.4.1 and can be found in Appendix A. The program takes five arguments $\alpha, \beta, \gamma, \tau$ and ρ and search for all possible realizable $(\tau\varphi\psi)$-triangles with $\rho < \varphi \leq \psi$ as follows: First the program goes through all possibilities of $n, \ell_1, \ell_2, \ell_3$. If the values satisfy (3.5), it tries to split $\ell_1\alpha + \ell_2\beta + \ell_3\gamma$ between φ and ψ. If two splitting provides the same pair (φ, ψ), we list it just once. Moreover, we test the condition (b) from Lemma 3.16 for $(\tau\varphi\psi)$ to further eliminate nonexistent triangles. As the last step, we use the spherical law of cosines (Lemma 3.16(c)) to compute the length of the edge x opposite to the angle τ and test if it can be obtained as a nonnegative integral combination of a, b, c. If x cannot be obtained as a combination of a, b, c, we list the best approximations of x by nonnegative integral combinations of a, b, c. In the output we round the numerical values to five decimal places.

For triangles $T = (\tau\varphi\psi)$ not excluded by the program we try to find some tiling of $\mathcal{T}$ by hand. We use Observation 3.30 to simplify the search. More precisely, in many cases it will be enough to find a tiling of a corner-filling tile whose two copies fill the whole τ-lune.

1) First we find realizable $(\alpha**)$-triangles for $\alpha = \pi/4$. The output of the program for $\alpha, \beta, \gamma, \alpha, 0$ is listed in Table 3.2. We immediately see, which triangles are candidates for being realizable. We refer to Figure 3.13 (left), that all candidates from Table 3.2 are indeed realizable. We note that triangles having $A'BD$, $A'BC$ in Figure 3.13 (left) are realizable by Observation 3.30 since two copies of ADE tile the α-lune.

Now we find all realizable $(\beta**)$-triangles. Since we already characterized all realizable $(\alpha**)$-triangles, we can assume that $\varphi, \psi > \alpha$. The output of the program for $\alpha, \beta, \gamma, \beta, \alpha$ can be found in Table 3.3. It can be easily checked that all candidates from Table 3.2 are realizable, we refer to Figure 3.13 (right). Note that triangles $B'AD, B'AC$ in Figure 3.13 are realizable by Observation 3.30.

2) We start with finding realizable $(\alpha**)$-triangles for $\alpha = \pi/5$. The output of

n	(ℓ_1, ℓ_2, ℓ_3)	(φ, ψ)	length of x	best approximation for x using a, b, c
2	(0,1,1)	(β, γ)	0.95532	$x = c$
3	(1,2,0)	$(\beta, \alpha+\beta)$	1.11184	$0.95532 \sim c < x < 2a \sim 1.23096$
4	(0,0,2)	(γ, γ)	1.0472	$0.95532 \sim c < x < 2a \sim 1.23096$
	(0,3,0)	$(\beta, 2\beta)$	1.23096	$x = 2a$
5	(1,1,1)	$(\gamma, \alpha+\beta)$	1.02671	$0.95532 \sim c < x < 2a \sim 1.23096$
		$(\beta, \alpha+\gamma)$	1.32931	$1.23096 \sim 2a < x < a+b \sim 1.40088$
6	(0,2,1)	$(\gamma, 2\beta)$	0.95532	$x = c$
		$(\beta, \beta+\gamma)$	1.41547	$1.40088 \sim a+b < x < 2b \sim 1.5708$
	(2,2,0)	$(\alpha+\beta, \alpha+\beta)$	0.91764	$0.7854 \sim b < x < c \sim 0.95532$
7	(1,0,2)	$(\gamma, \alpha+\gamma)$	0.7854	$x = b$
	(1,3,0)	$(\beta, \alpha+2\beta)$	1.49471	$1.40088 \sim a+b < x < 2b \sim 1.5708$
		$(\alpha+\beta, 2\beta)$	0.71907	$0.61548 \sim a < x < b \sim 0.7854$

Table 3.3: $(\beta\varphi\psi)$-triangles for $\alpha = \pi/4$, where $\varphi, \psi > \alpha$ and the approximate values of a, b, c are $a \sim 0.61548$, $b \sim 0.78540$, $c \sim 0.95532$.

the program for $\alpha, \beta, \gamma, \alpha, 0$ is listed in Table 3.4. It is not hard to check that all candidates from Table 3.4 are realizable, we refer to Figure 3.14 (left). Note that triangles $A'DE, A'DF, A'DB, A'CB$ are realizable by Observation 3.30, since two copies of ADH tile the α-lune.

Now we find all realizable $(\beta **)$-triangles. We may again assume that $\varphi, \psi > \alpha$. For all possible (β, φ, ψ)-triangles, where $\varphi, \psi > \alpha$, we refer to Table 3.5. It follows from Figure 3.14 left that all candidates from Table 3.5 are indeed realizable. We note that triangles $B'IG, B'DH, B'IA, B'AD$ and $B'AC$ are realizable by Observation 3.30, since two copies of BHI tile the β-lune.

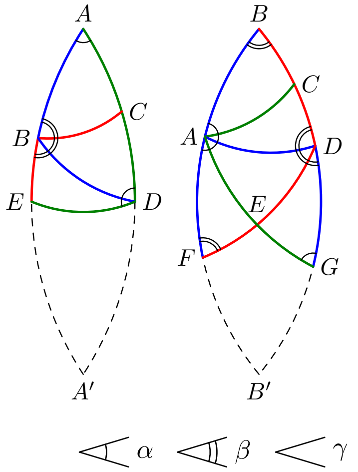
3) First we find all realizable $(\alpha **)$-triangles. The output of the program for $\alpha, \beta, \gamma, \alpha, 0$ is listed in Table 3.6. In order to prevent further confusion, we will always use 2β even if 3α is used in some tiling. Observe that all candidates from Table 3.6 are realizable, we refer to Figure 3.15 (left). Note that triangles $A'ED, A'BD$ and $A'BC$ are realizable by Observation 3.30, since two copies of AFD tile the α-lune. 3.7. Recall that we prefer 2β to 3α although they are equal.

It remains to find all realizable $(\beta**)$-triangles. For the output of the program, see Table 3.7. It might be surprising that one candidate from Table 3.7 is not realizable. More precisely, it is the $(\beta, \beta, 2\alpha+\beta)$-triangle. Indeed, assume the contrary. That is, $\mathcal{T} = (\beta, \beta, 2\alpha+\beta)$ is realizable with vertices B, U, V and the lengths of BU, UV equals $2b$. We may assume that $\mathcal{T}$ is corner-filling in the β-lune (B is a vertex of the lune). Since $\mathcal{T}$ is realizable and β is not a multiple of α, we may assume that some basic tile $\mathcal{T}_0 = (\alpha\beta\gamma)$ with vertices A, B, C is also a corner-filling tile of the β-lune. It follows that the edge BU of length $2b$ has to be composed from at least one edge a or c since the edge b of the basic tile is opposite to B. However, it can be easily checked (e.g. in Table 3.6) that $2b$ cannot be expressed as a nonnegative combination of a and c (the best approximation is $a + c < x < 3a$); a contradiction.

However, all the other candidates are realizable, we refer to Figure 3.15 (right). Triangles $B'AD$ and $B'AC$ are realizable by Observation 3.30,since two copies of BAH tile the β-lune.

□

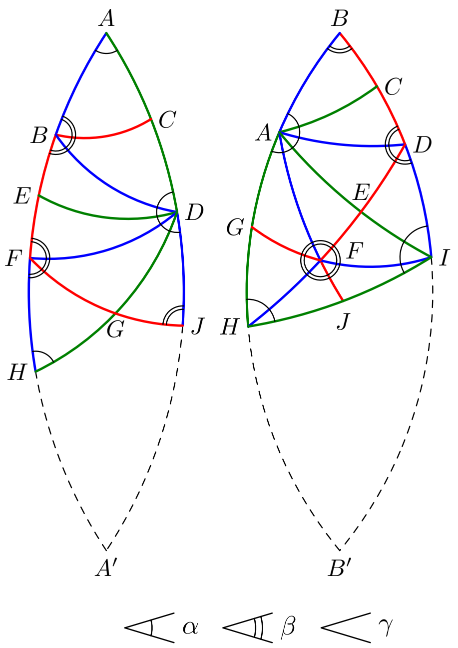
$(\alpha\varphi\psi)$-triangles	
$(\alpha, \alpha, 2\beta)$	ABD
(α, γ, γ)	ADE
$(\alpha, \beta, 3\alpha)$	$A'BD$
$(\alpha, \gamma, 2\beta)$	$A'BC$



$(\beta\varphi\psi)$-triangles	
(β, β, γ)	ABD
$(\beta, \beta, 2\beta)$	BDF
$(\beta, \gamma, 2\beta)$	$B'AD$
$(\beta, \gamma, 3\alpha)$	$B'AC$

Figure 3.13: Tiling of triangles with angle α (left) and β (right), where $\alpha = \frac{\pi}{4}$.

$(\alpha\varphi\psi)$-triangles	
$(\alpha, \alpha, 2\beta)$	ABD
$(\alpha, 2\alpha, \gamma)$	ADE
$(\alpha, \beta, 3\alpha)$	ADF
$(\alpha, \beta, 2\beta)$	AFJ
$(\alpha, \alpha, 4\alpha)$	ADH
$(\alpha, 2\alpha, 2\beta)$	$A'DF$
$(\alpha, \gamma, 3\alpha)$	$A'DE$
$(\alpha, \beta, 4\alpha)$	$A'DB$
$(\alpha, \gamma, 2\beta)$	$A'CB$



$(\beta\varphi\psi)$-triangles	
$(\beta, \beta, 2\alpha)$	ABD
$(\beta, 2\alpha, \gamma)$	BGI
$(\beta, 2\alpha, 3\alpha)$	BIH
$(\beta, \gamma, 3\alpha)$	$B'IG$
$(\beta, \beta, 4\alpha)$	$B'DH$
$(\beta, 2\alpha, 4\alpha)$	$B'IA$
$(\beta, 3\alpha, 2\beta)$	$B'AD$
$(\beta, \gamma, 4\alpha)$	$B'AC$

Figure 3.14: Tiling of triangles with angle α (left) and β (right), where $\alpha = \frac{\pi}{5}$.

n	(ℓ_1, ℓ_2, ℓ_3)	(φ, ψ)	length of x	best approximation for x using a, b, c
2	(1,2,0)	$(\beta, \alpha+\beta)$	0.4978	$0.36486 \sim a < x < b \sim 0.55357$
		$(\alpha, 2\beta)$	0.65236	$x = c$
3	(2,0,1)	$(\alpha, \alpha+\gamma)$	0.79357	$0.72973 \sim 2a < x < a+b \sim 0.91844$
		$(2\alpha, \gamma)$	0.55357	$x = b$
4	(3,1,0)	$(\beta, 3\alpha)$	0.65236	$x = c$
		$(\alpha, 2\alpha+\beta)$	0.91119	$0.72973 \sim 2a < x < a+b \sim 0.91844$
		$(2\alpha, \alpha+\beta)$	0.60731	$0.55357 \sim b < x < c \sim 0.65236$
6	(0,0,2)	(γ, γ)	0.62832	$0.55357 \sim b < x < c \sim 0.65236$
	(0,3,0)	$(\beta, 2\beta)$	0.72973	$x = 2a$
	(5,0,0)	$(\alpha, 4\alpha)$	1.10715	$x = 2b$
		$(2\alpha, 3\alpha)$	0.66185	$0.65236 \sim c < x < 2a \sim 0.72973$
7	(1,1,1)	$(\gamma, \alpha+\beta)$	0.6207	$0.55357 \sim b < x < c \sim 0.65236$
		$(\beta, \alpha+\gamma)$	0.74479	$0.72973 \sim 2a < x < a+b \sim 0.91844$
		$(\alpha, \beta+\gamma)$	1.19308	$1.10715 \sim 2b < x < b+c \sim 1.20593$
8	(2,2,0)	$(\beta, 2\alpha+\beta)$	0.74187	$0.72973 \sim 2a < x < a+b \sim 0.91844$
		$(\alpha, \alpha+2\beta)$	1.27389	$1.20593 \sim b+c < x < 2a+b \sim 1.2833$
		$(\alpha+\beta, \alpha+\beta)$	0.59348	$0.55357 \sim b < x < c \sim 0.65236$
		$(2\alpha, 2\beta)$	0.65236	$x = c$
9	(3,0,1)	$(\gamma, 3\alpha)$	0.55357	$x = b$
		$(\alpha, 2\alpha+\gamma)$	1.35103	$1.30472 \sim 2c < x < 2a+c \sim 1.38209$
		$(2\alpha, \alpha+\gamma)$	0.61739	$0.55357 \sim b < x < c \sim 0.65236$
10	(4,1,0)	$(\beta, 4\alpha)$	0.65236	$x = c$
		$(\alpha, 3\alpha+\beta)$	1.42562	$1.38209 \sim 2a+c < x < 4a \sim 1.45946$
		$(\alpha+\beta, 3\alpha)$	0.47458	$0.36486 \sim a < x < b \sim 0.55357$
		$(2\alpha, 2\alpha+\beta)$	0.5508	$0.36486 \sim a < x < b \sim 0.55357$
11	(0,2,1)	$(\gamma, 2\beta)$	0.36486	$x = a$
		$(\beta, \beta+\gamma)$	0.51894	$0.36486 \sim a < x < b \sim 0.55357$

Table 3.4: $(\alpha**)$-triangles for $\alpha = \pi/5$, $a \sim 0.36486, b \sim 0.55357, c \sim 0.65236$.

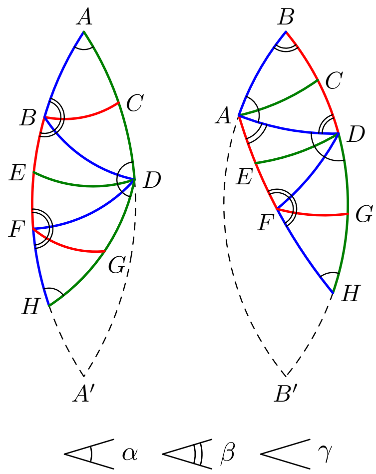
Lemma 3.32. *Let $\mathcal{T}_0 = (\alpha\beta\gamma)$, where $\alpha < \beta < \gamma$ and $\beta = \pi/3$. Then the α-edges and β-edges of S together form a subgraph isomorphic to one of the six graphs in Figure 3.16.*

Proof. The proof will be similar to the proof of Lemma 3.26.

Let $H = c(S)$ be the Coxeter diagram of S. Let $V(H) = \{u, v, w, x, y\}$ and let $H_\alpha = (V(G), E_\alpha), H_\beta = (V(G), E_\beta)$ and $H_{\alpha\beta} = (V(G), E_{\alpha\beta})$ be the subgraph of H formed by the α-edges, β-edges, and α and β-edges, respectively. Recall that H has at least two β-edges since is $(\alpha\beta\gamma)$-rich. By Lemma 3.26, H_α has two or three edges which form a subgraph isomorphic to $P_2 + P_2$ or $P_2 + P_3$, where $P_j + P_k$ is a disjoint union of paths with j and k vertices, respectively.

Suppose that $|E_\beta| = 2$. Then H_β is a matching by the completely same argument as in the case $|E_\alpha| = 2$ in Lemma 3.26. Indeed, in Lemma 3.26 we didn't use the assumptions $\alpha < \beta = \pi/3$. So let us assume that the β-edges $\{xu, yv\}$ form a matching. If H_α also forms a matching, then $H_{\alpha\beta}$ is isomorphic to an alternating 4-cycle, see Figure 3.16(a). Indeed, if $H_{\alpha\beta}$ forms an alternating path of length 5, then there are at most three $(\alpha\beta\gamma)$-triangles which is not possible. If H_α is isomorphic

$(\alpha\varphi\psi)$-triangles	
$(\alpha, \alpha, 2\beta)$	ABD
$(\alpha, 2\alpha, \gamma)$	ADE
$(\alpha, \beta, 2\beta)$	ADF
$(\alpha, \alpha, 4\alpha)$	ADH
$(\alpha, \gamma, \alpha + \beta)$	$A'ED$
$(\alpha, \beta, 2\alpha + \beta)$	$A'BD$
$(\alpha, \gamma, 2\beta)$	$A'BC$



$(\beta\varphi\psi)$-triangles	
$(\beta, \beta, 2\alpha)$	ABD
$(\beta, \alpha + \beta, 2\beta)$	$B'AD$
$(\beta, \gamma, 2\alpha + \beta)$	$B'AC$

Figure 3.15: Tiling of triangles with angle α (left) and β (right), where $\alpha = \frac{2\pi}{9}$.

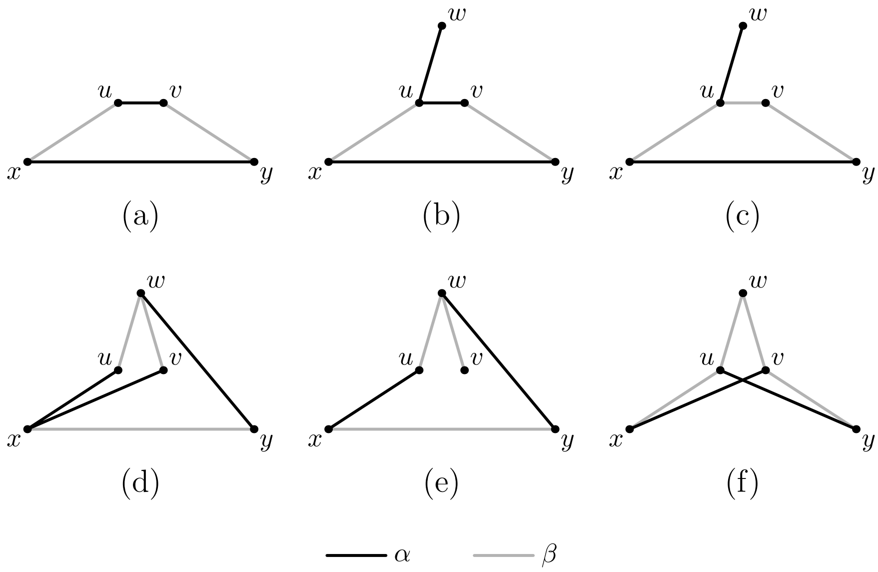


Figure 3.16: α-edges and β-edges of S for $\beta = \pi/3$

n	(ℓ_1, ℓ_2, ℓ_3)	(φ, ψ)	length of x	best approximation for x using a, b, c
2	(2,1,0)	$(\beta, 2\alpha)$	0.65236	$x = c$
4	(4,0,0)	$(2\alpha, 2\alpha)$	0.85216	$0.72973 \sim 2a < x < a + b \sim 0.91844$
5	(0,1,1)	(β, γ)	0.95532	$0.91844 \sim a + b < x < a + c \sim 1.01722$
6	(1,2,0)	$(\beta, \alpha + \beta)$	1.02412	$1.01722 \sim a + c < x < 3a \sim 1.09459$
7	(2,0,1)	$(2\alpha, \gamma)$	1.01722	$x = a + c$
8	(3,1,0)	$(\beta, 3\alpha)$	1.13794	$1.10715 \sim 2b < x < b + c \sim 1.20593$
		$(2\alpha, \alpha + \beta)$	1.05356	$1.01722 \sim a + c < x < 3a \sim 1.09459$
10	(0,0,2)	(γ, γ)	1.0472	$1.01722 \sim a + c < x < 3a \sim 1.09459$
	(0,3,0)	$(\beta, 2\beta)$	1.23096	$1.20593 \sim b + c < x < 2a + b \sim 1.2833$
	(5,0,0)	$(2\alpha, 3\alpha)$	1.10715	$x = 2b$
11	(1,1,1)	$(\gamma, \alpha + \beta)$	1.04401	$1.01722 \sim a + c < x < 3a \sim 1.09459$
		$(\beta, \alpha + \gamma)$	1.2722	$1.20593 \sim b + c < x < 2a + b \sim 1.2833$
12	(2,2,0)	$(\beta, 2\alpha + \beta)$	1.31082	$1.30472 \sim 2c < x < 2a + c \sim 1.38209$
		$(\alpha + \beta, \alpha + \beta)$	1.02796	$1.01722 \sim a + c < x < 3a \sim 1.09459$
		$(2\alpha, 2\beta)$	1.13794	$1.10715 \sim 2b < x < b + c \sim 1.20593$
13	(3,0,1)	$(\gamma, 3\alpha)$	1.01722	$x = a + c$
		$(2\alpha, \alpha + \gamma)$	1.1442	$1.10715 \sim 2b < x < b + c \sim 1.20593$
14	(4,1,0)	$(\beta, 4\alpha)$	1.38209	$x = 2a + c$
		$(\alpha + \beta, 3\alpha)$	0.97305	$0.91844 \sim a + b < x < a + c \sim 1.01722$
		$(2\alpha, 2\alpha + \beta)$	1.14298	$1.10715 \sim 2b < x < b + c \sim 1.20593$
15	(0,2,1)	$(\gamma, 2\beta)$	0.95532	$0.91844 \sim a + b < x < a + c \sim 1.01722$
		$(\beta, \beta + \gamma)$	1.41547	$1.38209 \sim 2a + c < x < 4a \sim 1.45946$
16	(1,0,2)	$(\gamma, \alpha + \gamma)$	0.90456	$0.72973 \sim 2a < x < a + b \sim 0.91844$
	(1,3,0)	$(\beta, \alpha + 2\beta)$	1.44777	$1.38209 \sim 2a + c < x < 4a \sim 1.45946$
		$(\alpha + \beta, 2\beta)$	0.87472	$0.72973 \sim 2a < x < a + b \sim 0.91844$
	(6,0,0)	$(2\alpha, 4\alpha)$	1.10715	$x = 2b$
		$(3\alpha, 3\alpha)$	0.85216	$0.72973 \sim 2a < x < a + b \sim 0.91844$
17	(2,1,1)	$(\gamma, 2\alpha + \beta)$	0.83279	$0.72973 \sim 2a < x < a + b \sim 0.91844$
		$(\beta, 2\alpha + \gamma)$	1.47923	$1.47201 \sim a + 2b < x < a + b + c \sim 1.5708$
		$(\alpha + \beta, \alpha + \gamma)$	0.79847	$0.72973 \sim 2a < x < a + b \sim 0.91844$
		$(2\alpha, \beta + \gamma)$	1.06021	$1.01722 \sim a + c < x < 3a \sim 1.09459$
18	(3,2,0)	$(\beta, 3\alpha + \beta)$	1.51008	$1.47201 \sim a + 2b < x < a + b + c \sim 1.5708$
		$(\alpha + \beta, 2\alpha + \beta)$	0.69014	$0.65236 \sim c < x < 2a \sim 0.72973$
		$(2\alpha, \alpha + 2\beta)$	0.97305	$0.91844 \sim a + b < x < a + c \sim 1.01722$
		$(3\alpha, 2\beta)$	0.65236	$x = c$
19	(4,0,1)	$(\gamma, 4\alpha)$	0.55357	$x = b$
		$(2\alpha, 2\alpha + \gamma)$	0.79357	$0.72973 \sim 2a < x < a + b \sim 0.91844$
		$(3\alpha, \alpha + \gamma)$	0.48235	$0.36486 \sim a < x < b \sim 0.55357$

Table 3.5: $(\beta\varphi\psi)$-triangles for $\alpha = \pi/5$, where $\varphi, \psi > \alpha$ and the approximate values of a, b, c are $a \sim 0.36486, b \sim 0.55357, c \sim 0.65236$.

to $P_2 + P_3$, there are three non-isomorphic placements of H_α. In the first case, the α-edges are $\{xy, uv, uw\}$, see Figure 3.16(b), in the second case $\{xy, uw, vw\}$ and in the third case $\{xy, yu, vw\}$. In the last case there are at most three $(\alpha\beta\gamma)$-triangles, i.e., yvw, yvu, yvx, even if all the remaining edges are γ-edges. It remains to solve the second case. In order to have four $(\alpha\beta\gamma)$-triangles, the edges $\{xw, yw, xv, yu\}$ have to be γ-edges. Due to the symmetry Φ simultaneously exchanging u with w and x with y, the $(\alpha\beta\gamma)$-triangles form only two orbits; a contradiction.

Suppose that $|E_\beta| = 3$. H_β cannot be isomorphic to the star $K_{1,3}$ by the same argument as in Lemma 3.26. Also, H_β cannot form a triangle, since the spherical triangle of type $(\beta\beta\beta) = (\pi/3, \pi/3, \pi/3)$ is not realizable by Lemma 3.16(a). If H_β is

n	(ℓ_1,ℓ_2,ℓ_3)	(φ,ψ)	length of x	best approximation for x using a,b,c
2	(1,2,0)	$(\beta,\alpha+\beta)$	0.64949	$0.48526\sim a<x<b\sim 0.67954$
		$(\alpha,2\beta)$	0.81199	$x=c$
	(4,0,0)	$(2\alpha,2\alpha)$	0.60772	$0.48526\sim a<x<b\sim 0.67954$
3	(2,0,1)	$(\alpha,\alpha+\gamma)$	0.98156	$0.97053\sim 2a<x<a+b\sim 1.1648$
		$(2\alpha,\gamma)$	0.67954	$x=b$
4	(0,0,2)	(γ,γ)	0.69813	$0.67954\sim b<x<c\sim 0.81199$
	(0,3,0)	$(\beta,2\beta)$	0.81199	$x=c$
	(3,1,0)	$(\alpha,2\alpha+\beta)$	1.12213	$0.97053\sim 2a<x<a+b\sim 1.1648$
		$(2\alpha,\alpha+\beta)$	0.70937	$0.67954\sim b<x<c\sim 0.81199$
5	(1,1,1)	$(\gamma,\alpha+\beta)$	0.67954	$x=b$
		$(\beta,\alpha+\gamma)$	0.83626	$0.81199\sim c<x<2a\sim 0.97053$
		$(\alpha,\beta+\gamma)$	1.24577	$1.1648\sim a+b<x<a+c\sim 1.29725$
6	(2,2,0)	$(\beta,2\alpha+\beta)$	0.81199	$x=c$
		$(\alpha,\alpha+2\beta)$	1.35908	$x=2b$
		$(\alpha+\beta,\alpha+\beta)$	0.60772	$0.48526\sim a<x<b\sim 0.67954$
		$(2\alpha,2\beta)$	0.64949	$0.48526\sim a<x<b\sim 0.67954$
7	(0,2,1)	$(\gamma,2\beta)$	0.48526	$x=a$
		$(\beta,\beta+\gamma)$	0.69336	$0.67954\sim b<x<c\sim 0.81199$
	(3,0,1)	$(\alpha,2\alpha+\gamma)$	1.46634	$1.45579\sim 3a<x<b+c\sim 1.49153$
		$(2\alpha,\alpha+\gamma)$	0.5207	$0.48526\sim a<x<b\sim 0.67954$

Table 3.6: $(\alpha**)$-triangles for $\alpha=2\pi/9$, $a\sim 0.48526, b\sim 0.67954, c\sim 0.81199$.

isomorphic to the path P_4, say, $E_\beta=\{xu,uv,vy\}$, then the edges xv and uy cannot have type α, since the spherical triangle of type $(\alpha\beta\beta)$ has smaller area than $\mathcal{T}_0$ and so it is not realizable. This implies that α-edges forms a subset of wx,wy,wu,wv,xy. Since neither $(\alpha\alpha\beta)$-triangle nor $(\alpha\alpha\alpha)$-triangle is realizable, it follows that H_α forms a matching; see Figure 3.16(c). The last option is that H_β forms a subgraph isomorphic to P_2+P_3, say, $E_\beta=\{uw,vw,xy\}$. Since the edge uv cannot be an α-edge ($(\alpha\beta\beta)$-triangle is not realizable), there are just three non-isomorphic placements of H_α. In the first case, H_α is isomorphic to P_2+P_3; see 3.16(d), in the second and third case to P_2+P_2. However, in the case when $E_\alpha=\{xu,yv\}$ there are, due to the symmetry Φ, at most two orbits of $(\alpha\beta\gamma)$-triangles (note that this case correspond to the case when $E_\alpha=\{xy,yu,vw\}$ and $E_\beta=\{xu,yv\}$, just the roles of α and β are exchanged). The last case is depicted in Figure 3.16(e).

Suppose that $|E_\beta|\geq 4$. If H_β contains a star $K_{1,4}$, then no other edge can be of type α, since the spherical triangle of type $(\alpha\beta\beta)$ has smaller area than $\mathcal{T}_0$. If H_β contains a "fork", say, $E_\beta\supseteq\{uv,vw,wx,wy\}$, then only two edges, ux and uy, can be of type α—a contradiction, since H_α is not a matching. If H_β contains a path P_5, say, $E_\beta\supseteq\{xu,uw,wv,vy\}$, then, using Lemma 3.26, only two edges, xv,uy can be of type α; see Figure 3.16(f). Since H_β cannot contain triangles, the last possibility is that H_β is isomorphic to the 4-cycle, say, $E_\beta=\{uv,vy,yx,xu\}$. All edges of type α must be incident with w, which is again a contradiction, since H_α is isomorphic to P_2+P_2 or P_2+P_3.

□

Lemma 3.33. *For $\alpha=\pi/4$ and $\alpha=\pi/5$, all $(\alpha\beta\gamma)$-rich Coxeter diagrams with five vertices whose all triangles of type $(\alpha**)$ and $(\beta**)$ are listed in Lemma 3.31 are*

n	(ℓ_1, ℓ_2, ℓ_3)	(φ, ψ)	length of x	best approximation for x using a, b, c
2	(2,1,0)	$(\beta, 2\alpha)$	0.81199	$x = c$
3	(0,1,1)	(β, γ)	0.95532	$0.81199 \sim c < x < 2a \sim 0.97053$
4	(1,2,0)	$(\beta, \alpha+\beta)$	1.06506	$0.97053 \sim 2a < x < a+b \sim 1.1648$
	(4,0,0)	$(2\alpha, 2\alpha)$	0.99245	$0.97053 \sim 2a < x < a+b \sim 1.1648$
5	(2,0,1)	$(2\alpha, \gamma)$	1.03827	$0.97053 \sim 2a < x < a+b \sim 1.1648$
6	(0,0,2)	(γ, γ)	1.0472	$0.97053 \sim 2a < x < a+b \sim 1.1648$
	(0,3,0)	$(\beta, 2\beta)$	1.23096	$1.1648 \sim a+b < x < a+c \sim 1.29725$
	(3,1,0)	$(2\alpha, \alpha+\beta)$	1.06506	$0.97053 \sim 2a < x < a+b \sim 1.1648$
7	(1,1,1)	$(\gamma, \alpha+\beta)$	1.03827	$0.97053 \sim 2a < x < a+b \sim 1.1648$
		$(\beta, \alpha+\gamma)$	1.29821	$1.29725 \sim a+c < x < 2b \sim 1.35908$
8	(2,2,0)	$(\beta, 2\alpha+\beta)$	1.35908	$x = 2b$
		$(\alpha+\beta, \alpha+\beta)$	0.99245	$0.97053 \sim 2a < x < a+b \sim 1.1648$
		$(2\alpha, 2\beta)$	1.06506	$0.97053 \sim 2a < x < a+b \sim 1.1648$
9	(0,2,1)	$(\gamma, 2\beta)$	0.95532	$0.81199 \sim c < x < 2a \sim 0.97053$
		$(\beta, \beta+\gamma)$	1.41547	$1.35908 \sim 2b < x < 3a \sim 1.45579$
	(3,0,1)	$(2\alpha, \alpha+\gamma)$	1.03	$0.97053 \sim 2a < x < a+b \sim 1.1648$
10	(1,0,2)	$(\gamma, \alpha+\gamma)$	0.85965	$0.81199 \sim c < x < 2a \sim 0.97053$
	(1,3,0)	$(\beta, \alpha+2\beta)$	1.46882	$1.45579 \sim 3a < x < b+c \sim 1.49153$
		$(\alpha+\beta, 2\beta)$	0.81199	$x = c$
	(4,1,0)	$(2\alpha, 2\alpha+\beta)$	0.95241	$0.81199 \sim c < x < 2a \sim 0.97053$
11	(2,1,1)	$(\gamma, 2\alpha+\beta)$	0.67954	$x = b$
		$(\beta, 2\alpha+\gamma)$	1.52026	$1.49153 \sim b+c < x < 2c \sim 1.62397$
		$(\alpha+\beta, \alpha+\gamma)$	0.6254	$0.48526 \sim a < x < b \sim 0.67954$
		$(2\alpha, \beta+\gamma)$	0.78127	$0.67954 \sim b < x < c \sim 0.81199$

Table 3.7: $(\beta\varphi\psi)$-triangles for $\alpha = 2\pi/9$, where $\varphi, \psi > \alpha$ and the approximate values of a, b, c are $a \sim 0.48526, b \sim 0.67954, c \sim 0.81199$.

shown in Figures 3.17 and 3.18. For $\alpha = 2\pi/9$, no $(\alpha\beta\gamma)$-rich Coxeter diagram with five vertices satisfying Lemma 3.31 exists.

Proof. Let H be a Coxeter diagram satisfying the assumptions of the lemma. Let $V(H) = \{u, v, w, x, y\}$ and let E_α, E_β be the set of α-edges and β-edges, respectively. By Lemma 3.32, we distinguish several cases, up to isomorphism. First of all we exclude case (f) when $E_\alpha = \{uy, xv\}, E_\beta = \{xu, uw, wv, vy\}$ for $\alpha \in \{\pi/4, \pi/5, 2\pi/9\}$. Since H is $(\alpha\beta\gamma)$-rich, the edges xy, uv have type γ. If the remaining edges xw, yw have the same type, then due to the symmetry Φ exchanging simultaneously u with v and x with y, the $(\alpha\beta\gamma)$-triangles form only at most three orbits; a contradiction. Let us assume that the edge xw has type φ. Since both triangles $(\beta\beta\varphi), (\alpha\beta\varphi)$ are realizable just for at most one φ for $\alpha \in \{\pi/4, \pi/5, 2\pi/9\}$, it follows that the edges xw, yw have the same type; a contradiction.

Therefore, by Lemma 3.32, we are left with cases (a)-(e) depicted in Figure 3.16, and we will continue separately for each value of α.

1) $\alpha = \pi/4$. Recall that by the spherical law of cosines (Lemma 3.16(c)), we have approximately, $a \sim 0.6155, b = \pi/4 \sim 0.7854$ and $c \sim 0.9553$. Note that the triangle of type $(\pi/2, \pi/2, 2\beta)$ is not realizable since its longest edge has length $2\pi/3 \sim 2.0944$, which cannot be obtained as a positive linear combination of a, b, c.

(a) $E_\alpha = \{uv, xy\}$, $E_\beta = \{xu, yv\}$. Since H is $(\alpha\beta\gamma)$-rich, the edges xv, uy have

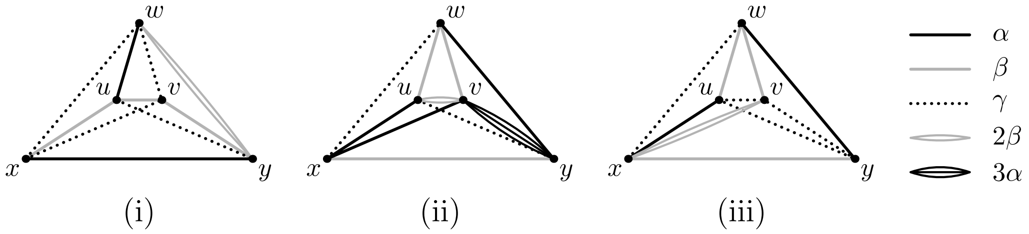


Figure 3.17: Coxeter diagrams for $\alpha = \frac{\pi}{4}$.

type γ. Since $(\alpha\gamma\gamma)$ and $(\alpha\gamma 2\beta)$ are the only realizable $(\alpha\varphi\psi)$-triangles for $\varphi, \psi \notin \{\alpha, \beta\}$, it follows that all edges incident with w are of type $\gamma, 2\beta$. If both the edges uw, vw are of type γ, then also wx, wy have type γ, since the $(\gamma, \gamma, 2\beta)$-triangle is not realizable. It follows that the triangle xuw has type $(\beta\gamma\gamma)$ and hence is not realizable; a contradiction. Thus one of the edges uw, vw is of type γ and the second one of type 2β. Without loss of generality let us assume that vw is of type 2β. Since the triangle $(\gamma, \gamma, 2\beta)$ is not realizable, it follows that the edge xw is of type 2β and the remaining edges uw, wy are of type γ. The diagram H has a symmetry group generated by the transpositions (x, v) and (u, y), hence it has at most two orbits; a contradiction. We conclude that the diagram (a) from Figure 3.16 can't be extended to H.

(b) $E_\alpha = \{uv, uw, xy\}$, $E_\beta = \{xu, yv\}$. As in the case (a), the edges xv, uy are of type γ. The types of edges vw and yw are uniquely determined: vw has type 2β and yw has type γ. Denote the type of xw by φ. Since there is no φ such that both triangles $(\alpha\beta\varphi)$ and $(\alpha\gamma\varphi)$ are realizable, it follows that the diagram (b) from Figure 3.16 also cannot be extended to H.

(c) $E_\alpha = \{uw, xy\}$, $E_\beta = \{xu, uv, vy\}$. In this case, since H is $(\alpha\beta\gamma)$-rich, the edges xw, wv, vx, uy have type γ. The type of the remaining edge wy is uniquely determined, since there is just one $\varphi = 2\beta$ for which both triangles $(\alpha\gamma\varphi)$, $(\beta\gamma\varphi)$ are realizable, see Figure 3.17(i).

(d) $E_\alpha = \{ux, xv, yw\}, E_\beta = \{uw, wv, xy\}$. The only possible type of uv is 2β. Since $(\gamma, \gamma, 2\beta)$ is not realizable, it follows that at most one edge from uy, vy has type γ. Since H contains at least two γ-edges, we can assume that uy and xw have type γ. The only possible type of vy is 3α, see Figure 3.17(ii).

(e) $E_\alpha = \{ux, yw\}, E_\beta = \{uw, wv, xy\}$. Since H is $(\alpha\beta\gamma)$-rich, the edges xw, uy have type γ. Denote the type of uv, xv, vy by φ, ψ, ω, respectively. Note that $\varphi, \psi, \omega \notin \{\alpha, \beta\}$. The triangles $(\beta\beta\varphi)$, $(\alpha\varphi\psi)$ and $(\beta\gamma\psi)$ are all realizable just for $\varphi = \gamma, \psi = 2\beta$, and the triangles $(\beta, 2\beta, \omega), (\alpha\beta\omega)$ just for $\omega = \gamma$, see Figure 3.17(iii).

Before continuing with $\alpha = \pi/5$, we exclude cases (c) and (e) for both $\alpha = \pi/5$ and $\alpha = 2\pi/9$, since the arguments are the same:

(c) $E_\alpha = \{uw, xy\}$, $E_\beta = \{xu, uv, vy\}$. Since $(\beta\beta\gamma)$ is not realizable, there are at most two $(\alpha\beta\gamma)$-triangles, i.e., uvw and xuw; a contradiction.

(e) $E_\alpha = \{ux, yw\}, E_\beta = \{uw, wv, xy\}$. Since H is $(\alpha\beta\gamma)$-rich, the edges xw, uy have type γ. Denote the type of uv, xv by φ, ψ, respectively. It follows $\varphi \in \{2\alpha, 4\alpha\}$. It is not hard to check that there is no ψ such that the triangles xuv, xvw which are of types $(\alpha\varphi\psi), (\beta\gamma\psi)$ are both realizable.

2) $\alpha = \pi/5$. Recall that by the spherical law of cosines (Lemma 3.16(c)), we have approximately, $a \sim 0.3649, b \sim 0.5536$ and $c \sim 0.6524$. Spherical triangles of type

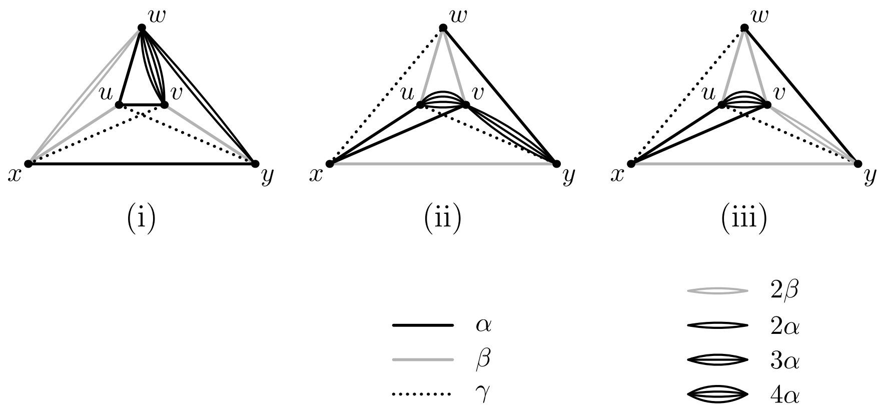


Figure 3.18: Coxeter diagrams for $\alpha = \frac{\pi}{5}$.

$(\pi/2, \pi/2, 4\alpha)$, $(\pi/2, \pi/2, 2\beta)$ and $(\pi/2, 2\alpha, 3\alpha)$ are not realizable since they have an edge of length $4\pi/5 \sim 2.5133$ or $2\pi/3 \sim 2.0944$ or ~ 1.2400, respectively, which cannot be obtained as a nonnegative linear combination of a, b, c. A spherical triangle of type $(\pi/2, 4\alpha, 4\alpha)$ does not exists by Lemma 3.16(b).

(a) $E_\alpha = \{uv, xy\}$, $E_\beta = \{xu, yv\}$. Since H is $(\alpha\beta\gamma)$-rich, the edges xv, uy have type γ. Let $\varphi, \psi, \omega, \tau$ be the types of wx, wu, wv, wy, respectively. Since the triangle $(\alpha\psi\omega)$ has to be realizable and since $\varphi, \psi, \omega, \tau \notin \{\alpha, \beta\}$, we have, up to symmetry, four options for (ψ, ω) : $(2\alpha, \gamma), (2\alpha, 2\beta), (\gamma, 3\alpha), (\gamma, 2\beta)$. Since there are just four $(\alpha\beta\gamma)$ triangles in H, H cannot be symmetric and hence $\{\varphi, \psi\} \neq \{\omega, \tau\}$. Since the triangles $(\beta\varphi\psi), (\beta\omega\tau), (\alpha\varphi\tau)$ are also realizable, we get, up to symmetry, that $\varphi = \gamma, \psi = 2\alpha, \omega = 2\beta, \tau = 3\alpha$ or $\varphi = \gamma, \psi = 2\alpha, \omega = \gamma, \tau = 3\alpha$. In the first case, the triangle xvw has type $(\gamma\varphi\omega) = (\gamma, \gamma, 2\beta)$, which is not realizable, in the second case, the triangle uyw has type $(pi/2, 2\alpha, 3\alpha)$ which is also not realizable.

(b) $E_\alpha = \{uv, uw, xy\}$, $E_\beta = \{xu, yv\}$. As in the case (a), the edges xv, uy are of type γ. Let φ, ψ, ω be the types of wv, wy, wx, respectively. Since $\varphi, \psi, \omega \notin \{\alpha, \beta\}$ and $(\alpha\alpha\varphi)$ has to be realizable, it is immediate that $\varphi \in \{2\beta, 4\alpha\}$. First, let us assume that $\varphi = 2\beta$. Since the triangles $(\alpha\gamma\psi), (\beta, 2\beta, \psi), (\alpha\beta\omega), (\alpha\psi\omega)$ are realizable, it follows $\psi = 3\alpha$ and $\omega = \gamma$. But now the triangle xvw has type $(\gamma, \gamma, 2\beta)$, which is not realizable; a contradiction. Thus we are left with the option $\varphi = 4\alpha$. Again, since the triangles $(\alpha\gamma\psi), (\beta, 4\alpha, \psi), (\alpha\beta\omega), (\alpha\psi\omega)$ are realizable, it follows $\psi = 2\alpha$ and $\omega \in \{\gamma, 2\beta\}$. If $\omega = \gamma$, the triangle xvw has type $(\gamma, \gamma, 4\alpha)$, which is not realizable, thus $\omega = 2\ beta$, see Figure 3.18(i).

(d) $E_\alpha = \{ux, xv, yw\}, E_\beta = \{uw, wv, xy\}$. The only possible type of uv is 4α. Since $(\gamma, \gamma, 4\alpha)$ is not realizable, it follows that at most one edge from uy, vy has type γ. Since H contains at least two γ-edges, we can assume that uy and xw have type γ. Since $(\gamma, 4\alpha, 4\alpha)$-triangle is not realizable, there are just two possible types of vy, namely 3α and 2β, see Figure 3.18(ii),(iii).

3) $\alpha = 2\pi/9$. Recall that by the spherical law of cosines (Lemma 3.16(c)), we have approximately, $a \sim 0.4853, b \sim 0.6780$ and $c \sim 0.8120$. Spherical triangle of type $(\pi/2, \pi/2, 2\beta)$ is not realizable since it have an edge of length $2\pi/3 \sim 2.0944$, which cannot be obtained as a nonnegative linear combination of a, b, c. The goal is to show that there is no H satisfying the required conditions.

(a) $E_\alpha = \{uv, xy\}$, $E_\beta = \{xu, yv\}$. Let $\varphi, \psi, \omega, \tau$ be the types of wx, wu, wv, wy, respectively. Since $\varphi, \psi, \omega, \tau \notin \{\alpha, \beta\}$, it is not hard to check that there are no $\varphi, \psi, \omega, \tau$ such that all triangles of types $(\alpha\psi\omega), (\beta\varphi\psi), (\beta\omega\tau), (\alpha\varphi\tau)$ are realizable.

(b) $E_\alpha = \{uv, uw, xy\}$, $E_\beta = \{xu, yv\}$. Since there is no realizable $(\beta, 4\alpha, *)$-triangle, the edges vw, yw has type $2\beta, \alpha+\beta$, respectively, by Lemma 3.31. It also follows that the edges xv, uy, xw has to be γ-edges, hence xvw has type $(\pi/2, \pi/2, 2\beta)$ which is a contradiction.

(d) $E_\alpha = \{ux, xv, yw\}, E_\beta = \{uw, wv, xy\}$. Again, using Lemma 3.31, we see that the triangles uvw and uvx cannot be both realizable. □

In order to finish the $(\alpha\beta\gamma)$ case it remains to show that simplices corresponding to Coxeter diagrams from Figures 3.17, 3.18 cannot exist. It is now easier than in Case (a) to compute all determinants of matrices corresponding to Coxeter diagrams from Figures 3.17 and 3.18, because we know precise values of all entries. We find that all the determinants are nonzero, which violates Fiedler's theorem.

For the sake of completeness, matrices B_1, B_2, B_3 which correspond to the diagrams in Figure 3.17 are listed below, the same holds for matrices C_1, C_2, C_3 which correspond to the diagrams in Figure 3.18. Again, the rows and columns of matrices are indexed by u, v, w, x, y (in this order).

$$B_1 = \begin{pmatrix} -1 & 0.5 & \sqrt{2}/2 & 0.5 & 0 \\ 0.5 & -1 & 0 & 0 & 0.5 \\ \sqrt{2}/2 & 0 & -1 & 0 & -0.5 \\ 0.5 & 0 & 0 & -1 & \sqrt{2}/2 \\ 0 & 0.5 & -0.5 & \sqrt{2}/2 & -1 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} -1 & -0.5 & 0.5 & \sqrt{2}/2 & 0 \\ -0.5 & -1 & 0.5 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0.5 & 0.5 & -1 & 0 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 & -1 & 0.5 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 & 0.5 & -1 \end{pmatrix}$$

$$B_3 = \begin{pmatrix} -1 & 0 & 0.5 & \sqrt{2}/2 & 0 \\ 0 & -1 & 0.5 & -0.5 & 0 \\ 0.5 & 0.5 & -1 & 0 & \sqrt{2}/2 \\ \sqrt{2}/2 & -0.5 & 0 & -1 & 0.5 \\ 0 & 0 & \sqrt{2}/2 & 0.5 & -1 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} -1 & \frac{1}{4}(\sqrt{5}+1) & \frac{1}{4}(\sqrt{5}+1) & 0.5 & 0 \\ \frac{1}{4}(\sqrt{5}+1) & -1 & -\frac{1}{4}(\sqrt{5}-1) & 0 & 0.5 \\ \frac{1}{4}(\sqrt{5}+1) & -\frac{1}{4}(\sqrt{5}-1) & -1 & -0.5 & \frac{1}{4}(\sqrt{5}-1) \\ 0.5 & 0 & -0.5 & -1 & \frac{1}{4}(\sqrt{5}+1) \\ 0 & 0.5 & \frac{1}{4}(\sqrt{5}-1) & \frac{1}{4}(\sqrt{5}+1) & -1 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} -1 & -\frac{1}{4}(\sqrt{5}-1) & 0.5 & \frac{1}{4}(\sqrt{5}+1) & 0 \\ -\frac{1}{4}(\sqrt{5}-1) & -1 & 0.5 & \frac{1}{4}(\sqrt{5}+1) & -\frac{1}{4}(\sqrt{5}+1) \\ 0.5 & 0.5 & -1 & 0 & \frac{1}{4}(\sqrt{5}+1) \\ \frac{1}{4}(\sqrt{5}+1) & \frac{1}{4}(\sqrt{5}+1) & 0 & -1 & 0.5 \\ 0 & -\frac{1}{4}(\sqrt{5}+1) & \frac{1}{4}(\sqrt{5}+1) & 0.5 & -1 \end{pmatrix}$$

$$C_3 = \begin{pmatrix} -1 & -\frac{1}{4}(\sqrt{5}-1) & 0.5 & \frac{1}{4}(\sqrt{5}+1) & 0 \\ -\frac{1}{4}(\sqrt{5}-1) & -1 & 0.5 & \frac{1}{4}(\sqrt{5}+1) & -0.5 \\ 0.5 & 0.5 & -1 & 0 & \frac{1}{4}(\sqrt{5}+1) \\ \frac{1}{4}(\sqrt{5}+1) & \frac{1}{4}(\sqrt{5}+1) & 0 & -1 & 0.5 \\ 0 & -0.5 & \frac{1}{4}(\sqrt{5}+1) & 0.5 & -1 \end{pmatrix}$$

The determinants rounded to two decimal places are:
$\det(B_1) = 0.06 \quad \det(B_2) = 0.13 \quad \det(B_3) = 0.21$
$\det(C_1) = 0.16 \quad \det(C_2) = 0.16 \quad \det(C_3) = 0.12$
This finishes the $(\alpha\beta\gamma)$ case and hence also the whole proof of Theorem 3.2.

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Appendix A

The program

```
def possible_tilings(u,v,w,lens,min_angle):

    def cos_law(a,b,c):
        z=(cos(c)+cos(a)*cos(b))/(sin(a)*sin(b));
        return numerical_approx(arccos(z));

    A=cos_law(w,v,u);
    B=cos_law(w,u,v);
    C=cos_law(u,v,w);

    def is_equal(x):
        var("a b c")
        min_error_below=10;meb_i=0;meb_j=0;meb_k=0;
        min_error_above=10;mea_i=0;mea_j=0;mea_k=0;
        for i in range(0,20):
             for j in range(0,20-i):
                for k in range (0,20-i-j):
                    t=x-i*A-j*B-k*C;
                    if (-0.00001<=t<=min_error_below):
                        min_error_below=t;meb_i=i;meb_j=j;meb_k=k;
                    if (-0.00001<=-t<=min_error_above):
                        min_error_above=-t;mea_i=i;mea_j=j;mea_k=k;
        if (min_error_below<0.00001):
            print round(x,5), "& $ x = ",
                  meb_i*a +meb_j*b + meb_k*c,
                  "$","!!!\\\\"
            return;
        print round(x,5), "& $", round(meb_i*A + meb_j*B + meb_k*C,5),
              "=", meb_i * a  + meb_j * b + meb_k * c, "<", "x", "<",
              mea_i*a + mea_j *b + mea_k*c, "=",
              round(mea_i*A + mea_j*B + mea_k*C,5), "$\\\\"

    def tries(x,y,z,tlens):
        visited = set();
        def split(n,x,y,z):
```

```
        var("alpha beta gamma");
        for i in range(0,x+1):
            for j in range(0,y+1):
                for k in range (0,z+1):
                    psi=i*u + j*v + k*w
                    phi=(x-i)*u + (y-j)*v + (z-k)*w
                    L = sorted([psi,phi,lens]);
                    if (0 < psi <= phi < pi
                      and L[1]+L[2] < pi+L[0]
                      and min_angle<psi
                      and (psi,phi) not in visited):
                          print n, "&", (x,y,z) ," &";
                          visited.add((psi,phi))
                          print "$(",(i*alpha)+j*beta+k*gamma,",",
                             (x-i)*alpha+(y-j)*beta+(z-k)*gamma,
                              ")$&";
                          is_equal(cos_law(psi,phi,lens));


    S = (x + y + z - pi);
    for d in range(2,2*tlens/S):
        for k in range(0,(d*S+pi-tlens)/x+1):
            for l in range(0,(d*S+pi-tlens-k*x)/y + 1):
                for m in range(0,(d*S+pi-tlens-k*x-l*y)/z + 1):
                    if (d*S + pi == k*x + l*y + m*z + tlens):
                        split(d,k,l,m)
tries(u,v,w,lens)
```