7^{TH} TUTORIAL ON RANDOMIZED ALGORITHMS

Eigenvalues of adjacency matrices and expanders II.

1. Show that a connected d-regular graph is bipartite iff the least eigenvalue of its adjacency matrix is $-d$.

2. Let A, B be two disjoint sets of vertices where $|A| = |B| = n$. We generate a random d-regular bipartite graphs for a fixed $d \geq 5$, by choosing uniformly at random d neighbors in B for each vertex in A (independently). We show that with constant positive probability each set $S \subseteq A$ of size $|S| \leq n/d$ has more than $d|S|/4$ neighbors.

Hint: define suitable indicators.

3. Let $G = (V, E)$ be a d-regular graph (so its largest eigenvalue is $\lambda_1 = d$). Assume that the absolute value of any other eigenvalue is at most λ for $0 \leq \lambda \leq \lambda_1$. Then for every $S \subseteq V$ with $|S| = \alpha \cdot n$, it holds that

$$|e(S, \overline{S}) - d \cdot (1 - \alpha) \cdot \alpha \cdot n| \leq \lambda \cdot \alpha \cdot (1 - \alpha) \cdot n \,,$$

where $e(S, \overline{S})$ is the number of edges between S and $\overline{S} = V \setminus S$.

Hint: use the following corollary of Courant-Fisher: Let u_1 be the eigenvector corresponding to λ_1. For any $x \in \mathbb{R}^n$ satisfying $\|x\| = 1$ and $x^T u_1 = 0$, it holds that $|x^T A x| \leq \lambda \cdot x^T x$. Now consider a vector that is negative on S and positive otherwise (or vice versa).

A side question: what is the meaning of $d \cdot (1 - \alpha) \cdot \alpha \cdot n$? To this end, consider a random d-regular graph, generated by picking d random neighbors of each vertex.