6^{TH} TUTORIAL ON RANDOMIZED ALGORITHMS

Eigenvalues of adjacency matrices

1. *Eigenvalues warm-up.* Let $A \in \mathbb{R}^{n\times n}$ be a matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Show that the matrix $A + dI_n$ has eigenvalues $d + \lambda_1, \ldots, d + \lambda_n$. What other basic properties of eigenvalues do you know?

2. Compute the eigenvalues and eigenvectors of the following graphs:

a) K_n, the complete graph on n vertices.

b) $K_{n,n}$, the complete bipartite graph with partites of size n each.

3. Show Courant-Fisher: Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix ($A^T = A$). Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be its eigenvalues. Show that $\lambda_1 = \max_{x\in\mathbb{R}^n, \|x\|=1} x^T Ax$.
(Similarly, $\lambda_n = \min_{x\in\mathbb{R}^n, \|x\|=1} x^T Ax$ and for example, $\lambda_2 = \max_{x\in\mathbb{R}^n, \|x\|=1, x^T u_1=0} x^T Ax$, where u_1 is the eigenvector corresponding to λ_1.)

4. Show that a connected d-regular graph is bipartite iff the least eigenvalue of its adjacency matrix is $-d$.

5. Bonus: Compute the eigenvalues of C_n, the cycle on n vertices.