1^{st} TUTORIAL ON RANDOMIZED ALGORITHMS

Intro and Markov chains

1. *Warm-up.* We want to fairly decide between two people, but we only have a biased coin, where heads appears with probability p. The value of p is unknown, except that $0 < p < 1$. However, we are allowed to flip the coin multiple times. How can we do it? What is the expected number of coin tosses?

2. *Envelopes.* You are presented with two sealed envelopes. There are k\$ in one of those and ℓ\$ in the other ($k, \ell \in \mathbb{N}$ but you do not know k, ℓ in advance). You may open an envelope and (based on what you see) decide to take this one or the other (without looking into both). Hint on the next page.

a) Is there a way how to walk away with the larger amount of money with probability strictly larger than 0.5?

b) What is the expected value you walk away with (in terms of k, ℓ)?

3. *Graph isomorphism.* You have seen an interactive proof of graph non-isomorphism during the lecture. What would be a trivial protocol for graph isomorphism? Can you come up with an "zero knowledge" interactive proof of graph isomorphism, that is, a protocol such that (informally) the verifier "learns nothing" about the isomorphism in case the two graphs are isomorphic?

In more detail, assuming that the verifier cannot solve the graph isomorphism problem and does not know the isomorphism beforehand, the verifier knows nothing more than that the two graphs are isomorphic at the end of the protocol. Such interactive proofs are called *zero knowledge.*

4. *Examples of Markov chains.* Come up with MCs with the following properties:

a) Create a MC that is irreducible.

b) Create a MC that is not irreducible.

c) Create a MC that is periodic.

d) Create a MC that is not periodic.

e) Compute a stationary distribution of the following MC:

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

f) Create a MC that has more stationary distributions.

g) A stochastic process that is *not* a Markov chain.

Some probability exercises:

5. *King*. The king of the land has exactly one sibling. What is the probability that the king has a brother? (Clarify all the assumptions you use — the problem has multiple variations, each leading to a different result!)

6. An urn contains a black and b white balls. We draw balls from it sequentially (without replacement). What is the probability that the first drawn ball is black? The second, third, ...?

7. Which is more likely:

a) Rolling a die four times and getting at least one six, or

b) Rolling a pair of dice 24 times and getting at least one double six?

8. *Bertrand's Paradox*. A circle is circumscribed around an equilateral triangle with side length 1. A random chord is selected within the circle. What is the probability that the length of the chord is greater than 1?

Hint for envelopes: We flip a coin until we get Tails. Let X denote the total number of flips (including the last one, i.e., $X \geq 1$). If our envelope contains m\$, we keep the envelope if $X < m$. What is the probability that we obtain the envelope with the higher amount?