Probabilistic techniques - tutorials

Classwork 6 – Markov chains

Definition 1. Let $r^t_{i,j} = \Pr[X_t = j \text{ and } X_s \neq j \text{ for all } 0 < s < t | X_0 = i]$. A state i is *recurrent* if $\sum_{t\geq 1} r^t_{i,i} = 1$ and it is *transient* otherwise.

Definition 2. A communicating class C is *closed* if for all $i \in C$ it holds that if j is accessible from i, then j is in C as well.

Definition 3. A communicating class C is *recurrent* if it contains a recurrent state. HW problem: If $i \in C$ is recurrent, then every $j \in C$ is recurrent.

1. Let $\{X_i\}_{i=0}^{\infty}$ be a (homogeneous) Markov chain with the transition matrix P. Prove the following.
   (a) A state i is transient if and only if $\sum_{n\geq 0} P^n_{i,i} < \infty$. That is, i is recurrent if and only if $\sum_{n\geq 0} P^n_{i,i} = \infty$.
   (b) A state i is recurrent if and only if $\Pr[X_n = i \text{ for infinitely many } n | X_0 = i] = 1$.
2. Let i be a recurrent state and assume that j is accessible from i, then i is accessible from j. In particular recurrent communicating classes are closed.
3. Every finite closed communicating class is recurrent.