Probabilistic techniques - tutorials

Classwork 6 – Markov chains

Definition 1. Let $r_{i,j}^t = \Pr[X_t = j \text{ and } X_s \neq j \text{ for all } 0 < s < t | X_0 = i]$. A state i is *recurrent* if $\sum_{t\geq 1} r_{i,i}^t = 1$ and it is *transient* otherwise.

Definition 2. A communicating class C is *closed* if for all $i \in C$ it holds that if j is accessible from i, then j is in C as well.

Definition 3. A communicating class C is *recurrent* if it contains a recurrent state. HW problem: If $i \in C$ is recurrent, then every $j \in C$ is recurrent.

1. Let $\{X_i\}_{i=0}^{\infty}$ be a (homogeneous) Markov chain with the transition matrix P. Prove the following.

   (a) A state i is transient if and only if $\sum_{n\geq 0} P_{i,i}^n < \infty$. That is, i is recurrent if and only if $\sum_{n\geq 0} P_{i,i}^n = \infty$.

   Solution:

   i. Let $p_i = \Pr[\text{There exists } t \geq 1 \text{ such that } X_t = i | X_0 = i] = \sum_{t\geq 1} r_{i,i}^t$.

   ii. Let N_i be the random variable that denotes the number of visits to i.

   iii. Compute expectation of N_i: $Pr[N_i = n | X_0 = i] = p_i^{n-1}(1 - p_i)$, then

   $$E[N_i | X_0 = i] = \sum_{n\geq 1} np_i^{n-1}(1 - p_i) = (1 - p_i)\frac{d}{dp}\sum_{n\geq 1} p_i^n = \frac{1}{1 - p_i}.$$

   iv. We can compute the same expectation with indicator random variables:

   $$E[N_i | X_0 = i] = E[\sum_{n\geq 1} I_{X_n = i} | X_0 = i] = \sum_{n\geq 1} \Pr[X_n = i | X_0 = i] = \sum_{n\geq 1} P_{i,i}^n.$$

   (b) A state i is recurrent if and only if

   $$\Pr[X_n = i \text{ for infinitely many } n | X_0 = i] = 1.$$

   Conversely, the state i is transient if and only if

   $$\Pr[X_n = i \text{ for infinitely many } n | X_0 = i] = 0.$$

   Solution:

   i. $B_k = \{X_n = i \text{ for at least } k \text{ different values of } n \in \mathbb{N}\}$

   ii. Then $\Pr[B_k | X_0 = i] = p_i^k$.

   iii. $B_1 \supset B_2 \supset B_3 \supset \cdots$.

   iv. $\Pr[X_n = i \text{ for intinitely many } n | X_0 = i] = \Pr[\lim_k B_k | X_0 = i] = \lim_k \Pr[B_k | X_0] = \lim_k p_i^k$.

   v. This last one is 1 iff $p_i^k = 1$ if i is recurrent, and 0 otherwise.

2. Let i be a recurrent state and assume that j is accessible from i, then i is accessible from j. In particular recurrent communicating classes are closed.

   Solution:

   (a) Since j is accessible from i, we can pick smallest k such that such that $P_{i,j}^k > 0$.

   (b) If i is not accessible from j, then $p_i = \Pr[\text{returning to } i | X_0 = i] \leq 1 - P_{i,j}^k < 1$.

(c) This is a contradiction with i being recurrent.

3. Every finite closed communicating class is recurrent.

   Solution:

   (a) Let C be a closed communicating class with finitely many elements.

   (b) A chain starting in i stays in C forever and since C is finite, there must be at least one state $j \in C$ which is visited infinitely often with positive probability, i.e. $\Pr[X_n = j \text{ for infinitely many } n] > 0$.

   (c) Since i and j are in the same communicating class, there exists $m \in \mathbb{N}$ so that $P^m_{j,i} > 0$.

   (d) From the inequality

   $$\Pr[X_n = j \text{ for infinitely many } n | X_0 = j] \geq P^m_{j,i} \Pr[X_n = j \text{ for infinitely many } n | X_0 = i] > 0$$

   it follows that state j is recurrent by Exercise 1(b).

   (e) The class C is then recurrent because it contains at least one recurrent state, namely j.