Probabilistic techniques - tutorials

Classwork 5 – Lovász local lemma and Chernoff bound

1. Prove that, for every integer $d > 1$, there is a finite $c(d)$ such that the edges of any bipartite graph with maximum degree d in which every cycle has at least $c(d)$ edges can be colored by $d + 1$ colors so that there are no two adjacent edges with the same color and there is no two-colored cycle.

2. Let m and k be two positive integers satisfying

   $$e(m(m-1)+1)k(1-\frac{1}{k})^m \leq 1.$$

   Then, for any set S of m real numbers, there is a k-coloring of $\mathbb{R}$ such that each translation $x + S$, for $x \in \mathbb{R}$, is multicolored. That is $c(x + S) = [k]$.

3. Let σ be a uniformly random permutation of $[n] = \{1, \ldots, n\}$. Denote $X = |\{i \in [n] : (\forall j < i)\sigma(j) < \sigma(i)\}|$. Prove that for every $\epsilon > 0$ it holds that

   $$\lim_{n\to\infty} \Pr[(1-\epsilon)H_n < X < (1+\epsilon)H_n] = 1,$$

   where $H_n = \sum_{i=1}^{n} \frac{1}{i}$.