Probabilistic techniques - tutorials

Classwork 5 – Lovász local lemma and Chernoff bound

1. Prove that, for every integer $d > 1$, there is a finite $c(d)$ such that the edges of any bipartite graph with maximum degree d in which every cycle has at least $c(d)$ edges can be colored by $d + 1$ colors so that there are no two adjacent edges with the same color and there is no two-colored cycle.

   Solution: Note that the graph has a proper edge-coloring by at most d colors. Indeed, we can extend the graph to a d-regular bipartite graph (such that we don't care about lengths of cycles). The edges of such a graph can be partitioned into d perfect matchings (it has a perfect matching by Hall's theorem and the remaining graph is again regular, so we can iterate). We can use this partition as colors, so also the edges of the original graph can be colored by at most d colors.

   The idea is then to randomly recolor each edge by the new color $d+1$ and use LLL to show that with non-zero probability this recoloring with satisfy to required properties. This is similar to Theorem 4 from `https://people.math.ethz.ch/~sudakovb/acyclic.pdf`.

2. Let m and k be two positive integers satisfying

$$e(m(m-1)+1)k(1-\frac{1}{k})^m \leq 1.$$

   Then, for any set S of m real numbers, there is a k-coloring of $\mathbb{R}$ such that each translation $x + S$, for $x \in \mathbb{R}$, is multicolored. That is $c(x + S) = [k]$.

   Solution:

   (a) We first show the claim for a finite set $X \subseteq \mathbb{R}$ of translations.

   (b) Set $Y = \cup_{x\in X}(x + S)$ and let $c : Y \to \{1, 2, \ldots, k\}$ be a random k-coloring of Y obtained by choosing for each $y \in Y$ the color $c(y)$ uniformly and independently.

   (c) For each $x \in X$, let A_x be the event in which $x + S$ is not multicolored (with respect to c). Clearly, $\Pr[A_x] = \Pr[\cup_i \text{ missing color } i] \leq k(1-1/k)^m$, because the probability of the event missing color i is $(1 - 1/k)^m$.

   (d) Moreover, each event A_x is mutually independent of all the other events A_x but those for which $(x + S) \cap (x\ + S) \neq \emptyset$.

   (e) There are at most $m(m-1)$ such events, because there is a unique x' such that for some $s, s' \in S, s \neq s'$, we have $x + s = x' + s'$. Then $e(m(m-1)+1)k(1-1/k)^m \leq 1$ and we can apply LLL.

   (f) We can now prove the existence of a coloring of the set of all reals with the desired properties, by a standard compactness argument. Since the discrete space with k points is (trivially) compact, Tikhonov's Theorem implies that an arbitrary product of such spaces is compact. In particular, the space of all functions from $\mathbb{R}$ to $\{1, 2, \ldots, k\}$, with the usual product topology, is compact. That is, the open set are those that have finite number of coordinates been not the whole space.

   (g) In this space, for every fixed $x \in \mathbb{R}$, the set C_x of all colorings c, such that $x + S$ is multicolored, is closed. Because we are specifiyng the value on a finite set of points.

   (h) As we proved above, the intersection of any finite number of sets C_x is nonempty. It thus follows, by compactness, that the intersection of all sets C_x is nonempty. Any coloring in this intersection has the properties we want.

3. Let σ be a uniformly random permutation of $[n] = \{1, \ldots, n\}$. Denote $X = |\{i \in [n] : (\forall j < i)\sigma(j) < \sigma(i)\}|$. Prove that for every $\epsilon > 0$ it holds that

$$\lim_{n\to\infty} \Pr[(1-\epsilon)H_n < X < (1+\epsilon)H_n] = 1,$$

where $H_n = \sum_{i=1}^{n} \frac{1}{i}$.

Solution:

(a) Set $X_i = 1_{(\forall j<i)\sigma(j)<\sigma(i)}$ then $X = \sum_{i=1}^{n} X_i$.

(b) On the other hand

$$E[X_i] = \Pr[\forall j < i : \sigma(j) < \sigma(i)] = (i-1)!\frac{n!}{i!}\frac{1}{n!} = \frac{1}{i},$$

we can permute all the elements greater than i first, then the position of i is completely determined as it has to be the greatest. Finally we have to permute the elementes strickly lower than i, and we have $i-1$ of these. Then $E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} 1/i = H_n$.

(c) Let us check that X_i are independent. Consider a subset $I = \{i_1 < \cdots < i_k\} \subset [n]$, we want to see that $\Pr[\cap_{i\in I} X_i = 1] = \prod_{i\in I} \Pr[X_i = 1]$. This is similar to the previous argument. To compute $\Pr[\cap_{i\in I} X_i = 1]$, first order all the greater elements than i_k, $n!/i_k!$, then order all the elements between $i_{k-1} < i_k$, $(i_k - 1)!/i_{k-1}!$ and so on.

(d) Then by Chernoff $\Pr[|X - H_n| \geq \epsilon H_n] < 2e^{-\frac{\epsilon^2 H_n}{3}}$, which goes to 0 since H_n goes to ∞.