Probabilistic techniques - tutorials

Classwork 3 – The method of alternation

1. Denote by $R(\cdot,\cdot)$ the Ramsey numbers. Fix $k \in \mathbb{N}$. Prove that for any integer n, it holds that $R(k,k) > n - \binom{n}{k}2^{1-\binom{k}{2}}$. Can you write the expression on the right hand side in terms of k?

   Solution: (Alon-Spencer Theorem 3.1.1) Take a random coloring of K_n and remove one vertex from each monochromatic copy of K_k. There is a coloring with at most $\binom{n}{k}2^{1-\binom{k}{2}}$ copies. Thus, there is a graph with at least $n - \binom{n}{k}2^{1-\binom{k}{2}}$ vertices without monochromatic copy of K_k.

   Choosing n as a suitable function of k, you can prove

   $$R(k,k) > \frac{1}{e}(1+o(1))k2^{k/2}.$$

2. Fix a graph F and define $\mathrm{ex}(n,F)$ to be the maximal number of edges of an n-vertex graph that does not contain F as a subgraph. Show that for any F on $k > 2$ vertices with at least $2k-3$ edges it holds $\mathrm{ex}(n,F) = \Omega(n^{3/2})$.

   Solution: Consider $G(n,p)$. To ensure that the graph does not contain F, we remove one edge from each k tuple containing at least $2k-3$ edges. The expected number of such copies is at most $n^k p^{2k-3}$. Let X stand for the number of edges of the resulting graph. Then, by setting $p = cn^{-1/2}$, we have

   $$\mathbb{E}[X] \geq \binom{n}{2}cn^{-1/2} - n^k c^{2k-3} n^{-1/2(2k-3)} \geq \frac{c}{3}n^{3/2} - c^{2k-3}n^{3/2} = \left(\frac{1}{3} - c^{2k-4}\right)cn^{3/2}.$$

   We can set $c = 1/3$.

3. A dominating set of an undirected graph $G = (V,E)$ is a set $U \subseteq V$ such that every vertex $v \in V \setminus U$ has at least one neighbor in U. Let $G = (V,E)$ be a graph on n vertices, with minimum degree $\delta > 1$. Then G has a dominating set of at most $n\frac{1+\ln(\delta+1)}{\delta+1}$ vertices.

   Solution: Let D be a dominating set created as follows. Put $v \in A$ independently with probability p. Then, let B be the set of vertices from $V \setminus A$ that are not dominated by A. Set $D = A \cup B$. We have $\mathbb{E}[A] = np$ and $\mathbb{E}[B] \leq n(1-p)^{\delta+1}$. Thus the expected size of D is

   $$\mathbb{E}[D] = \mathbb{E}[A] + \mathbb{E}[B] \leq np + n(1-p)^{\delta+1} \leq np + ne^{-p(\delta+1)}.$$

   By setting $p = \frac{\ln(\delta+1)}{\delta+1}$, we get the result in expectation. Thus, there exists such a set D.

4. Let $G = (V,E)$ be a graph on n vertices with minimum degree $\delta > 1$. Prove that there is a partition of V into two disjoint sets A and B such that $|A| \leq \mathcal{O}(n\frac{ln(\delta)}{\delta})$ and each vertex in B has at least one neighbor in A and at least one neighbor in B.

   Solution: We say that a vertex $v \in V$ is over-dominated by a set $S \subset V, v \notin S$ if $N(v) \subset S$. First put $v \in X$ independently with probability p. Let Y be the vertices over-dominated by X and set $A_0 = X \cup Y$. It remains to solve not-dominated vertices, let us call them Z_0. Consecutively for each vertex $v \in Z_i$ we proceed as follows: let $A_{i+1} = A_i \cup \{v\}$ if it does not create a new over-dominated vertex. Otherwise, let $A_{i+1} = A_i \cup \{u\}$ instead, where u is the neighbor of u to be over-dominated. Update the set Z_i to Z_{i+1}.

   Observe that $|Z_i|$ always decreases, thus at the end of the process, we have $|A| \leq |X| + |Y| + |Z_0|$. Therefore, in expectation

   $$\mathbb{E}[|A|] \leq np + n(1-p)p^{\delta} + n(1-p)^{\delta+1} \leq n(2p + e^{-p(\delta+1)}),$$

which for $p = \frac{\ln(\delta)}{\delta+1}$ simplifies as follows

$$\mathbb{E}[|A|] \leq n\left(2 \cdot \frac{\ln(\delta)}{\delta+1} + \frac{1}{\delta}\right) \leq n\left(2 \cdot \frac{\ln(\delta)}{\delta} + 2 \cdot \frac{\ln(\delta)}{\delta}\right) = 4n\frac{\ln(\delta)}{\delta}.$$

5. Define the number $m(n)$ as follows: given any n-uniform hypergraph $H = (V, E)$ less than $m(n)$ edges, there exists a two-coloring of V such that no edge is monochromatic. Show that $m(n) = \Omega(2^n(n/\ln(n))^{1/2})$.

   Solution: (Alon-Spencer Theorem 3.1.1) We are proving the following statement: if there is $p \in [0,1]$ for which $k(1-p)^n + k^2p < 1$, then $m(n) > 2^{n-1}k$. The result is then obtained by setting $p = \frac{\ln(n/k)}{n}$ and $k = c(n/\ln(n))^{1/2}$, where $c < \sqrt{2}$.

   Fix H = (V, E) with $m = 2^{n-1}k$ edges and p satisfying the condition. For each vertex $v \in V$, let x_v be a uniformly chosen label form $[0,1]$. Use the following greedy algorithm: process vertices by their labels and color them blue unless they are the last vertices on an edge, then use red. Observe then only red edge may arise and only in the case that all its vertices, in particular its first vertex, are the last vertices of some other edge. Call an ordered pair of edges (e, f) a *conflicting* pair it the last vertex of e is the first vertex of f. We proceed to show that with positive probability there are no such pairs.

   Split the interval $[0,1]$ into three parts $L = [0, \frac{1-p}{2}), M = [\frac{1-p}{2}, \frac{1+p}{2}), R = [\frac{1+p}{2}, 1]$. The probability of an conflicting pair (e, f) with $e \subset L$ or $f \subset R$ is clearly bounded by $2k2^{n-1}(\frac{1-p}{2})^n = k(1-p)^n$.

   Otherwise, the vertex $v = e \cap f$ lies in M, this is with probability p. Moreover, we have $e \leq v \leq f$ with probability $x_v^{n-1}(1-x_v)^{n-1} \leq (1/4)^{n-1}$. Furthermore, there are at most $k^2 4^{n-1}$ possible ordered conflicting pairs.

   Thus, the probability of a conflicting pair is at most

   $$k(1-p)^n + p \cdot (1/4)^{n-1} \cdot k^2 4^{n-1} = k(1-p)^n + pk^2$$

   which is less than 1 by our assumption. Thus there is a positive probability of not observing a conflicting pair, i.e. of finding a proper coloring.