Probabilistic techniques - tutorials

Classwork 2 – Linearity of expectation

1. Compute the expected number of fixed points of a random permutation on $[n]$.

   Solution: Let I_i be the indicator that $\pi(i) = i$. Then

   $$\mathbb{E}[\#\text{ fixed points}] = \mathbb{E}\left[\sum_{i=1}^{n} I_i\right] = \sum_{i=1}^{n} \mathbb{E}[I_i] = n \cdot \Pr[I_1 = 1] = n \cdot \frac{1}{n} = 1.$$

2. Show that there is a two coloring of the edges of K_n with at most $\binom{n}{a}2^{1-\binom{a}{2}}$ monochromatic K_a.

   Solution: For $S \in \binom{V}{a}$ let be I_S be the indicator that $K_n[S]$ is monochromatic. Then

   $$\mathbb{E}[\#\text{ monochromatic } K_a] = \mathbb{E}\left[\sum_{S} I_S\right] = \sum_{S} \mathbb{E}[I_S] = \binom{n}{a}\Pr[K_n[S] \text{ monochromatic}] = \binom{n}{a}2^{1-\binom{a}{2}}.$$

   Thus, there is a coloring with as few as $\binom{n}{a}2^{1-\binom{a}{2}}$ monochromatic copies of K_a.

3. Show that there is a two coloring of edges of $K_{n,m}$ with at most $\binom{n}{a}\binom{m}{b}2^{1-ab}$ monochromatic $K_{a,b}$.

   Solution: As above.

4. Let $A, B \in \binom{n}{k}$ be chosen independently uniformly at random. Compute $\mathbb{E}[|A \cap B|]$.

   Solution: Let $I_{i,A}$ be the indicator that $i \in A$, similarly for B. Then

   $$\mathbb{E}[|A \cap B|] = \mathbb{E}\left[\sum_{i=1}^{n} I_{i,A} \cdot I_{i,B}\right] = n\,\mathbb{E}[I_{i,A} \cdot I_{i,B}] = n\,\mathbb{E}[I_{i,A}]\,\mathbb{E}[I_{i,B}] = n\frac{k^2}{n^2} = \frac{k^2}{n}.$$

5. Let M be an $n \times n$ matrix with entries uniformly independent chosen from $\{-1, 1\}$. Determine $\mathbb{E}[\det(M)]$.

   Solution: We have

   $$\mathbb{E}[\det(M)] = \mathbb{E}\left[\sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_{i} M_{i,\pi(i)}\right] = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_{i} \mathbb{E}[M_{i,\pi(i)}] = 0.$$

6. Let $n \geq 2$, $H = (V, E)$ an n-uniform hypergraph with $|E| = 4^{n-1}$ edges. Show that there is a coloring of V by four colors such that no edge is monochromatic.

   Solution: For $e \in E$ let be I_e be the indicator that e is monochromatic. Then

   $$\mathbb{E}[\#\text{ monochromatic edges}] = \mathbb{E}\left[\sum_{e} I_e\right] = |E|\Pr[e \text{ monochromatic}] = 4^{n-1} \cdot 4^{1-n} = 1.$$

   Since there are events attaining greater value than 1 (e.g. monochromatic V), there are events fewer monochromatic edges than 1, i.e. 0.

Information about tutorials `https://iuuk.mff.cuni.cz/~tung/teaching/probtech-winter2526`