Probabilistic techniques - tutorials

Classwork 1 – Basics

By classical probability space we denote the probability space $(\Omega, 2^\Omega, \Pr)$ where Ω is a finite set and $\Pr[A] = |A|/|\Omega|$. We define $[n] = \{1, \dots, n\}$. An n-uniform hypergraph H is a tuple (V, E) where the elements of E are subset of V of size n.

1. You flip a coin 6 times. Compute the probability of the event "There is an even number of heads or there are exactly 3 heads and 3 tails".

2. Prove that there exist a constant $c > 0$ such that for every integers n and m holds that if $n > cm \log m$, then a random mapping $[n] \to [m]$ is surjective with probability at least 0.99.

3. A *tournament* is a graph obtained by taking a clique and orienting each edge in exactly one direction.

   Prove the following statement: If $\binom{n}{k}(1 - 2^{-k})^{n-k} < 1$, then there exists a tournament $T = (V, E)$ with $|V| = n$ such that for every subset $U \subset V$ of size k there exists $v \in V \setminus U$ such that $(v, u) \in E$ for all $u \in U$.

4. Find an example of three non-empty, non-certain events A, B and C in classical probability space that are not independent, but it holds that $\Pr[A \cap B \cap C] = \Pr[A]\Pr[B]\Pr[C]$. What is the minimum size of a probability space with these events?

5. Let $\{(A_i, B_i) \colon i = 1, \dots, h\}$ be pairs of subsets of an arbitrary set such that $|A_i| = k$, $|B_i| = \ell$, $A_i \cap B_i = \emptyset$ and $A_i \cap B_j \neq \emptyset$ for $i \neq j$. Show that $h \leq \binom{k+\ell}{k}$.