Probabilistic techniques - tutorials

Classwork 1 – Basics

By classical probability space we denote the probability space $(\Omega, 2^\Omega, \Pr)$ where Ω is a finite set and $\Pr[A] = |A|/|\Omega|$. We define $[n] = \{1, \ldots, n\}$. An n-uniform hypergraph H is a tuple (V, E) where the elements of E are subset of V of size n.

1. You flip a coin 6 times. Compute the probability of the event "There is an even number of heads or there are exactly 3 heads and 3 tails".

   Solution:

$$\frac{|A|}{|\Omega|} = \frac{\binom{6}{0} + \binom{6}{2} + \binom{6}{4} + \binom{6}{6} + \binom{6}{3}}{2^6} = \frac{1 + 15 + 15 + 1 + 20}{64} = \frac{52}{64} = \frac{13}{16}.$$

2. Prove that there exist a constant $c > 0$ such that for every integers n and m holds that if $n > cm \log m$, then a random mapping $[n] \to [m]$ is surjective with probability at least 0.99.

   Solution: Let f be the random function and $A_i, i \in [m]$ be the event that "$i \in f[n]$". It holds

$$\Pr[\text{surjective}] = \Pr\left[\bigcap_i A_i\right] = 1 - \Pr\left[\bigcup_i \overline{A_i}\right] \geq 1 - m\left(1 - \frac{1}{m}\right)^n \geq 1 - me^{-n/m} > 1 - me^{-(cm \log m)/m} = 1 - m^{1-c}.$$

3. A *tournament* is a graph obtained by taking a clique and orienting each edge in exactly one direction.

   Prove the following statement: If $\binom{n}{k}(1 - 2^{-k})^{n-k} < 1$, then there exists a tournament $T = (V, E)$ with $|V| = n$ such that for every subset $U \subset V$ of size k there exists $v \in V \setminus U$ such that $(v, u) \in E$ for all $u \in U$.

   Solution: Let T be a random tournament, i.e. the probability of one particular direction of an edge is $1/2$. Let $A_{v,U}$ be the event that "$(v, u) \in E$ for all $u \in U$". Observe that $A_{v,U}$ and $A_{v',U}$ for $v \neq v'$ are independent. It holds

$$\Pr[\text{bad choice}] = \Pr\left[\bigcup_U \bigcap_v \overline{A_{v,U}}\right] \leq \sum_U \Pr\left[\bigcap_v \overline{A_{v,U}}\right] = \binom{n}{k} \prod_v \Pr\left[\overline{A_{v,U}}\right] = \binom{n}{k}(1 - 2^{-k})^{n-k} < 1.$$

4. Find an example of three non-empty, non-certain events A, B and C in classical probability space that are not independent, but it holds that $\Pr[A \cap B \cap C] = \Pr[A]\Pr[B]\Pr[C]$. What is the minimum size of a probability space with these events?

   Solution: Take an ordinary dice $D \in \{1, \ldots, 6\}$ as the probability space with events "$D \leq 3$", "$D \geq 3$", "D odd". No smaller probability space with these events exists due to divisibility problems, e.g. if $|\Omega| = 5$, then $\Pr[A \cap B \cap C] = n/5$ while $\Pr[A]\Pr[B]\Pr[C] = abc/75$.

5. Let $\{(A_i, B_i) : i = 1, \ldots, h\}$ be pairs of subsets of an arbitrary set such that $|A_i| = k$, $|B_i| = \ell$, $A_i \cap B_i = \emptyset$ and $A_i \cap B_j \neq \emptyset$ for $i \neq j$. Show that $h \leq \binom{k+\ell}{k}$.

   Solution: (Alon-Spencer Theorem 1.3.3) Set $X = \bigcup_i (A_i \cup B_i)$ and consider a random ordering of its elements. Let E_i be the event that "all the elements of A_i precede all elements of B_i". Observe that $\Pr[E_i] = 1/\binom{k+l}{k}$ and that E_i and E_j for $i \neq j$ are disjoint. Then

$$1 \geq \Pr\left[\bigcup_i E_i\right] = \sum_i \Pr[E_i] = h/\binom{k+l}{k}.$$

   This is sharp, as witnessed by the family $\{(A_i, [k+l] \setminus A_i) : A_i \in \binom{[k+l]}{k}\}$.

Information about tutorials `https://iuuk.mff.cuni.cz/~tung/teaching/probtech-winter2526`