# Probabilistic techniques - tutorials

Classwork 6 – Markov chains

**Definition 1.** Let $r_{i,j}^t = \Pr[X_t = j \text{ and } X_s \neq j \text{ for all } 0 < s < t | X_0 = i]$. A state $i$ is *recurrent* if $\sum_{t\geq 1} r_{i,i}^t = 1$ and it is *transient* otherwise.

**Definition 2.** A communicating class $C$ is *closed* if for all $i \in C$ it holds that if $j$ is accessible from $i$, then $j$ is in $C$ as well.

**Definition 3.** A communicating class $C$ is *recurrent* if it contains a recurrent state. HW problem: If $i \in C$ is recurrent, then every $j \in C$ is recurrent.

1. Let $\{X_i\}_{i=0}^{\infty}$ be a (homogeneous) Markov chain with the transition matrix $P$. Prove the following.
   (a) A state $i$ is transient if and only if $\sum_{n\geq 0} P_{i,i}^n < \infty$. That is, $i$ is recurrent if and only if $\sum_{n\geq 0} P_{i,i}^n = \infty$.
   (b) A state $i$ is recurrent if and only if $\Pr[X_n = i \text{ for infinitely many } n | X_0 = i] = 1$.
2. Let $i$ be a recurrent state and assume that $j$ is accessible from $i$, then $i$ is accessible from $j$. In particular recurrent communicating classes are closed.
3. Every finite closed communicating class is recurrent.