# Probabilistic techniques - tutorials

## Classwork 5 – Lovász local lemma and Chernoff bound

1. Prove that, for every integer $d > 1$, there is a finite $c(d)$ such that the edges of any bipartite graph with maximum degree $d$ in which every cycle has at least $c(d)$ edges can be colored by $d+1$ colors so that there are no two adjacent edges with the same color and there is no two-colored cycle.

   *Solution:*

   (a) We first prove that $G$ has an edge-coloring $\alpha$ by $d$ colors. Observe that $G$ can be embedded to a $d$-regular bipartite graph $H$. By Hall's theorem, $H$ has a perfect matching. We can remove the perfect matching to get a $d-1$-regular graph and iterate.

   (b) Observe that each edge $e$ is at most in $d-1$ two-colored cycles in $\alpha$ (determined by the color $\alpha(e)$ and another color).

   (c) We break each two-colored cycle $C$ into disjoint paths $P_1^C, \dots, P_k^C$ of length $[c(d), 2 \cdot c(d))$ (if $|C| < 2 \cdot c(d)$, we have just one path). We call these paths *fragments*. We fix this partition for the rest of the proof.

   (d) We uniformly select an edge from each fragment $P_i^C$ at random, independently for each path, and color this edge with the $(d+1)$-th color. We denote the new coloring by $\beta$.

   (e) Let $B_e$ be the *bad* event that "$\beta(e) = d+1$ and either there is an edge $e'$ adjacent to $e$ with $\beta(e') = d+1$ or $e''$ joined by an edge to $e$ with $\beta(e'') = d+1$." In other words "$\beta(e) = d+1$ and the color $d+1$ also appears in the (open) second edge-neighborhood of $e$."

   (f) Observe that if neither of the events $B_e$ occurs, the coloring $\beta$ is as required. The first part with the edge $e'$ implies that $\beta$ is a proper edge-coloring. The part with the edge $e''$ implies that there are no two-colored cycles. That is, each two-colored cycle in $\alpha$ now has an edge of color $d+1$ by construction. Thus, the only possible two-color cycles has $d+1$ as one of the colors. However, if an edge $e$ gets the color $d+1$ then by the absence of $d+1$ in the second neighborhood, we cannot get a two-color cycle.

   (g) We want to apply Lovász Local Lemma to show $\Pr[\bigcap_e \overline{B}_e] > 0$. To do so, we will estimate $\Pr[B_e]$ and compute the dependency degree among the events $\{B_e : e \in E(G)\}$.

   (h) Let us denote by $N_2(e)$ the open second edge-neighborhood of $e$, by $P_e$ the set of fragments containing the edge $e$. We have $|P_e| \leq d-1$ from above. Moreover, denote by $A_{e,P}$ the event that $\beta(e) = d+1$ by the fragment $P \in P_e$, and by $A_e$ the event that $\beta(e) = d+1$. We have

   $$\Pr[B_e] = \Pr[\bigcup_{f \in N_2(e)} A_e \cap A_f] = \Pr[\bigcup_{f \in N_2(e)} \bigcup_{\substack{P \in P_e \\ Q \in P_f}} A_{e,P} \cap A_{f,Q}].$$

   If the fragments $P$ and $Q$ are different, the events $A_{e,P}$ and $A_{f,Q}$ are independent. Otherwise, the probability of their intersection is 0 (assuming $e \neq f$, which we have here). Thus, we have $\Pr[A_{e,P} \cap A_{e,Q}] \leq \Pr[A_{e,P}] \cdot \Pr[A_{e,Q}]$ in both cases.

   $$\Pr[B_e] = \Pr[\bigcup_{f \in N_2(e)} \bigcup_{\substack{P \in P_e \\ Q \in P_f}} A_{e,P} \cap A_{f,Q}] \leq \sum_{f \in N_2(e)} \sum_{\substack{P \in P_e \\ Q \in P_f}} \Pr[A_{e,P}] \cdot \Pr[A_{e,Q}]$$
   $$\leq 2((d-1)^2 + d - 1) \cdot (d-1)^2 \cdot \frac{1}{c(d)^2} \leq \frac{4d^4}{c(d)^2}.$$

(i) To compute the degree, observe that the events $B_e$ and $B_f$ are independent if there are no two edges $e' \in N_2(e)$ and $f' \in N_2(f)$ that lie in the same fragment. Thus, the degree is at most

$$\underbrace{2((d-1)^2+d-1)}_{(i)} \cdot \underbrace{d \cdot 2c(d)}_{(ii)} \cdot \underbrace{2((d-1)^2+d-1)}_{(iii)} \leq 32d^5c(d),$$

where (i) counts the choices of $e'$, (ii) counts the choices of $f'$ sharing a fragment with $e'$ (there is at most $d$ fragments, each of which of length at most $2c(d)$), and (iii) counts the choices of $f$ such that $f \in N_2(f')$, or equivalently $f' \in N_2(f)$.

(j) If $e \cdot \frac{4d^4}{c(d)^2} \cdot 32d^5c(d) = e\frac{128d^9}{c(d)} \leq 1$, we are done.

2. Let $m$ and $k$ be two positive integers satisfying

$$e(m(m-1)+1)k(1-\frac{1}{k})^m \leq 1.$$

Then, for any set $S$ of $m$ real numbers, there is a $k$-coloring of $\mathbb{R}$ such that each translation $x+S$, for $x \in \mathbb{R}$, is multicolored. That is $c(x+S) = [k]$.

*Solution:*

(a) We first show the claim for $x$ in a finite subset $X \subseteq \mathbb{R}$.

(b) Set $Y = \cup_{x \in X}(x+S)$ and let $c : Y \to \{1, 2, \ldots, k\}$ be a random $k$-coloring of $Y$ obtained by choosing for each $y \in Y$ the color $c(y)$ uniformly and independently.

(c) For each $x \in X$, let $A_x$ be the event in which $x+S$ is not multicolored (with respect to $c$). Clearly, $\Pr[A_x] = \Pr[\cup_i \text{ missing color } i] \leq k(1-1/k)^m$, because the probability of the event missing color $i$ is $(1-1/k)^m$.

(d) Moreover, each event $A_x$ is mutually independent of all the other events $A_{x'}$ but those for which $(x+S) \cap (x'+S) \neq \emptyset$.

(e) There are at most $m(m-1)$ such events, because there is a unique $x$ such that for $s, s' \in S$, $x+s=s'$. Then $e(m(m-1)+1)k(1-1/k)^m \leq 1$ and we can apply LLL.

(f) We can now prove the existence of a coloring of the set of all reals with the desired properties, by a standard compactness argument. Since the discrete space with $k$ points is (trivially) compact, Tikhonov's Theorem implies that an arbitrary product of such spaces is compact. In particular, the space of all functions from $\mathbb{R}$ to $\{1, 2, \ldots, k\}$, with the usual product topology, is compact. That is, the open set are those that have finite number of coordinates been not the whole space.

(g) In this space, for every fixed $x \in \mathbb{R}$, the set $C_x$ of all colorings $c$, such that $x+S$ is multicolored, is closed. Because we are specifiyng the value on a finite set of points.

(h) As we proved above, the intersection of any finite number of sets $C_x$ is nonempty. It thus follows, by compactness, that the intersection of all sets $C_x$ is nonempty. Any coloring in this intersection has the properties we want.

3. Let $\sigma$ be a uniformly random permutation of $[n] = \{1, \ldots, n\}$. Denote $X = |\{i \in [n] : (\forall j < i)\sigma(j) < \sigma(i)\}|$. Prove that for every $\epsilon > 0$ it holds that

$$\lim_{n\to\infty} \Pr[(1-\epsilon)H_n < X < (1+\epsilon)H_n] = 1,$$

where $H_n = \sum_{i=1}^n \frac{1}{i}$.

*Solution:*

(a) Set $X_i = 1_{(\forall j<i)\sigma(j)<\sigma(i)}$ then $X = \sum_{i=1}^n X_i$.

(b) On the other hand

$$E[X_i] = \Pr[\forall j < i : \sigma(j) < \sigma(i)] = (i-1)!\frac{n!}{i!}\frac{1}{n!} = \frac{1}{i},$$

we can permute all the elements greater than $i$ first, then the position of $i$ is completely determined as it has to be the greatest. Finally we have to permute the elementes strickly lower than $i$, and we have $i-1$ of these. Then $E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} 1/i = H_n$.

(c) Let check that $X_i$ are independent. Consider a subset $I = \{i_1 < \cdots < i_k\} \subset [n]$, we want to see that $\Pr[\cap_{i\in I} X_i = 1] = \prod_{i\in I} \Pr[X_i = 1]$. This is similar to the previous argument. $\Pr[\cap_{i\in I} X_i = 1]$ first order all the greater elements than $i_k$, $n!/i_k!$, then order all the elements between $i_{k-1} < i_k$, $(i_k - 1)!/i_{k-1}!$ and so on.

(d) Then by Chernoff $\Pr[|X - H_n| \geq \epsilon H_n] < 2e^{-\frac{\epsilon^2 H_n}{3}}$, which goes to 0 since $H_n$ goes to $\infty$.