Probabilistic techniques - tutorials

Classwork 3 – The method of alternation

1. Denote by $R(\cdot,\cdot)$ the Ramsey numbers. Fix $k \in \mathbb{N}$. Prove that for any integer n, it holds that $R(k,k) > n - \binom{n}{k} 2^{1-\binom{k}{2}}$. Can you write the expression on the right hand side in terms of k?

2. Fix a graph F and define $\mathrm{ex}(n,F)$ to be the maximal number of edges of an n-vertex graph that does not contain F as a subgraph. Show that for any F on $k > 2$ vertices with at least $2k-3$ edges it holds $\mathrm{ex}(n,F) = \Omega(n^{3/2})$.

3. A dominating set of an undirected graph $G = (V,E)$ is a set $U \subseteq V$ such that every vertex $v \in V \setminus U$ has at least one neighbor in U. Let $G = (V,E)$ be a graph on n vertices, with minimum degree $\delta > 1$. Then G has a dominating set of at most $n\frac{1+\ln(\delta+1)}{\delta+1}$ vertices.

4. Let $G = (V,E)$ be a graph on n vertices with minimum degree $\delta > 1$. Prove that there is a partition of V into two disjoint sets A and B such that $|A| \leq \mathcal{O}(n\frac{ln(\delta)}{\delta})$ and each vertex in B has at least one neighbor in A and at least one neighbor in B.

5. Define the number $m(n)$ as follows: given any n-uniform hypergraph $H = (V,E)$ less than $m(n)$ edges, there exists a two-coloring of V such that no edge is monochromatic. Show that $m(n) = \Omega(2^n(n/\ln(n))^{1/2})$.