1

. $Pr[\text{even number of } H] = \sum_{k\in\{0,2,4,6\}} Pr[\text{there is } k \text{ heads}] = \sum_{k\in\{0,2,4,6\}} \binom{6}{k}\left(\frac{1}{2}\right)^6 = \left(\frac{1}{2}\right)^6\left(\binom{6}{0}+\binom{6}{2}+\binom{6}{4}+\binom{6}{6}\right)$

$= \left(\frac{1}{2}\right)^6 (1+15+15+1) = \frac{1}{2}$

. $Pr[\text{exactly } 3 \text{ heads } 3 \text{ tails}] = \binom{6}{3}\left(\frac{1}{2}\right)^6 = \frac{20}{64} = \frac{5}{16}$

. $Pr[\text{task}] = \frac{1}{2} + \frac{5}{16} = \frac{13}{16}$

2

. want to apply union bound

. for $i \in [m]$ let A_i : i is not in $rng(f)$ for a randomly selected f

. $Pr_f[f \text{ is surjective}] = 1 - Pr[f \text{ not surjective}] = 1 - Pr\left[\bigcup_{i=1}^{m} A_i\right]$

\# mappings ignoring i

. $Pr[A_i] = \frac{(m-1)^n}{m^n} = \left(\frac{m-1}{m}\right)^n = \left(1-\frac{1}{m}\right)^n \leq e^{-\frac{n}{m}}$

all mappings — apply $1+x \leq e^x \ \forall x \in \mathbb{R}$

. $Pr\left[\bigcup_{i=1}^{m} A_i\right] \underset{\text{union bound}}{\leq} \sum_{i=1}^{m} Pr(A_i) \leq m \cdot e^{-\frac{n}{m}} \leq m \cdot e^{-\frac{c m \log m}{m}} = m^1 \cdot m^{-c} = m^{1-c}$

$\rightarrow Pr[\text{not surjective}] \leq m^{1-c}$, can always pick c large enough

---

3:

. a gr is a tournament if it is a result of taking a clique and orienting each edge in exactly one direction

. pick a random tournament, i.e. orient each edge of a clique randomly

. let $A_{U,v}$ for $U \subseteq V$, $v \in V \setminus U$ be the event that "all edges between v and U point from v out"

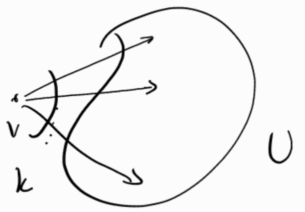
- what we want is to show that $\Pr\left[\bigcap_{\substack{U \subseteq V \\ |U|=k}} \bigcup_{v \in V \setminus U} A_{U,v}\right] > 0$

- equivalently we want $\Pr\left[\overline{\bigcap_{\substack{U \subseteq V \\ |U|=k}} \bigcup_{v \in V \setminus U} A_{U,v}}\right] < 1$

$= \Pr\left[\bigcup_{\substack{U \subseteq V \\ |U|=k}} \bigcap_{v \in V \setminus U} \overline{A_{U,v}}\right]$

- when does $\overline{A_{U,v}}$ occur? When there is and edge from U pointing to v

- $\Pr[\overline{A}_{U,v}] = 1 - 2^{-k}$ random orientation of T

$\Pr\Big[$  $\Big] = 2^{-k}$ ... all edges point to U

union bound

- $\Pr\left[\bigcup_{\substack{U \subseteq V \\ |U|=k}} \bigcap_{v \in V \setminus U} \overline{A_{U,v}}\right] \leq \binom{n}{k} (1 - 2^{-k})^{n-k} < 1$ by statement

over all subsets $U \subseteq V$ of size k

for all vertices $V \setminus U$, $|V \setminus U| = n-k$

$\square$

4

- take a dice w/ events $D \leq 3$, $D \geq 3$, D odd
- no smaller space exists because of divisibility issues
  - if $|\Omega| = 5$ then $\Pr[A \cap B \cap C] = \frac{?}{2^5}$ but $\Pr[A]\Pr[B]\Pr[C] = \frac{?}{5^3}$

5.

- assume A_i's, B_i's are subsets of some $X = [n]$
- let us define new set pairs $\mathcal{F} = \{(C_i, D_i)_{i=1}^h\}$ in the following way
  - take any $\binom{X}{k+l}$ (i.e. $(k+l)$-tuple of elems of X) and let C_i be the k smallest elems of the tuple and D_i the l largest elems of the tuple
- for every C_i, let e_i be the maximal element of C_i $(i, j \in [h])$
- take two e_i, e_j defined here and suppose $e_i \le e_j$
  (*)
  - then every elem of C_i is at most e_i and every elem of D_j is strictly more than e_i

  $\to$ either $C_i \cap D_j = \emptyset$ or $C_j \cap D_i = \emptyset$
- define $\mathcal{F}^1, \mathcal{F}^2, \dots, \mathcal{F}^{n!}$ by taking the construction of $\mathcal{F}$ above but permuting elems of X
  - we denote elems of $\mathcal{F}^i = \{(C_1^i, D_1^i), (C_2^i, D_2^i), \dots\}$
  - $\mathcal{F}_j^i = (C_j^i, D_j^i)$
- from (*) we have $\forall p \in [n!]$ either $C_i^p \cap D_j^p = \emptyset$ or $C_j^p \cap D_i^p = \emptyset$ (this is just extension of notatio)

- now I show a proof that has no probability because I belive it to be more transparent
  - but it uses magical double counting ¯_(ツ)_/¯

- $\forall q \in [h]$ $p \in [n!]$ we count two different ways of pairs $(\mathcal{F}_p, (A_q, B_q))$ such that $C_i^p = A_q$ and $D_i^p = B_q$
  - so we fix a list $\mathcal{F}_p$ and we look for pairs in $\mathcal{F}_p$ which look like (A_q, B_q)
- fix p first
- suppose $C_i^p = A_q$, $D_i^p = B_q$ and simultaneously $C_j^p = A_r$, $D_j^p = B_r$ for $1 \le p \overset{!}{<} q \le h$
- but then $C_i^p \cap D_j^p = \emptyset$ and $C_j^p \cap D_i^p = \emptyset$, which contradicts (*)
  - do you see why?
  - recall $C_i^p < D_i^p$
    $C_j^p < D_j^p$

$\rightarrow$ so for every p there is one (A_q, B_q) that can exist
  - and there's $n!$ such pairs

- now let's fix $q \in [h]$ ... we fix A_q, B_q and we search for $\mathcal{F}_j^{i}$'s
- such $\mathcal{F}_j^i$ is obtained by permuting X into A_q and B_q

$\rightarrow$ there's $\binom{|X|}{k+l}$ sets C_i^p, D_i^p that can be permuted into A_q B_q

- how many permutations are there?

$$\left.\begin{array}{ll} C_i^p \to A_q & k! \\ \& & \\ D_i^p \to B_q & l! \\ \& & \\ \text{the rest we don't care} & (n-k-l)! \end{array}\right\} \quad k!\,l!\,(n-k-l))$$

- so there are

$$\binom{n}{k+l} k!\, l!\, (n-k-l)!$$ $\mathcal{F}_j'$'s in total for each selection of σ

- putting everything together

$$h \binom{n}{k+l} k!\, l!\, (n-k-l)! \leq n! \xrightarrow{\text{arithmetic autopilot}} \text{done}$$

- tight example $\{ (A, [k+l] \setminus A) : \forall A \subseteq [k+l],\ |A| = k \}$