Systems of equations, Analytic geometry

Tung Anh Vu

September 25, 2023

KAM

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

Contact: tung@kam.mff.cuni.cz

Systems of equations

One variable, one equation

Types of equations:

- *Linear*:
$$6x + 3 = 0.$$
- *Quadratic*:
$$2x^2 + 3x + 1 = 0.$$
- *Cubic*:
$$x^3 - 5x^2 - 2x + 24 = 0.$$
- *Quartic*, *quintic*,...

Systems of equations

One variable, one equation

Types of equations:

- *Linear*:
$$6x + 3 = 0.$$
- *Quadratic*:
$$2x^2 + 3x + 1 = 0.$$
- *Cubic*:
$$x^3 - 5x^2 - 2x + 24 = 0.$$
- *Quartic*, *quintic*,...
- Can have 0, 1, multiple, or infinitely many solutions.

Solving linear equations

Linear equations can have either:

- zero solutions
$$7x + 3 = 7x + 2,$$
- one solution
$$6x + 9 = x - 6,$$
- infinitely many solutions
$$5x + 3 - 4x = 3 + x.$$

Solving quadratic equations

General form

$$ax^2 + bx + c = 0,$$

where $b, c \in \mathbb{R}$ and $a \in \mathbb{R} \setminus \{0\}$.

Solving quadratic equations

General form

$$ax^2 + bx + c = 0,$$

where $b, c \in \mathbb{R}$ and $a \in \mathbb{R} \setminus \{0\}$.

Example

Given $2x^2 + 3x + 1 = 0$, we have $a = 2,\ b = 3,\ c = 1$.

Solving quadratic equations: quadratic formula

Quadratic formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solving quadratic equations: quadratic formula

Quadratic formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Task

Solve $2x^2 + 3x + 1 = 0$ using the quadratic formula.

Solving polynomial equations: by factoring

Rational zero test

Each **rational solution** x of a polynomial equation is of the form $\frac{p}{q}$ where

- p is a factor of the constant term, and
- q is a factor of the leading term.

Solving polynomial equations: by factoring

Rational zero test

Each **rational solution** x of a polynomial equation is of the form $\frac{p}{q}$ where

- p is a factor of the constant term, and
- q is a factor of the leading term.

Tasks

Solve the following by factoring:

- $x^2 + 2x - 15 = 0$,
- $x^3 - 7x + 6 = 0$.

Multivariate equations

One equation

- Over reals $\mathbb{R}$ has generally infinitely many solutions.
- Over integers $\mathbb{Z}$ may be extremely difficult to solve.
  - E.g., Fermat's last theorem.

Two equations, two variables

Number of solutions

- Can have 0, 1, multiple or infinitely many solutions.

Two equations, two variables

Number of solutions

- Can have 0, 1, multiple or infinitely many solutions.
- If the equation is linear, then each equation defines a line in $\mathbb{R}^2$.

Two equations, two variables

Number of solutions

- Can have 0, 1, multiple or infinitely many solutions.
- If the equation is linear, then each equation defines a line in $\mathbb{R}^2$.
- And the solution is the intersection of those lines.

Two equations, two variables

Method of substitution

1. Solve

$$\begin{aligned} x^2 + 4x - y &= 7 \\ 2x - y &= -1 \end{aligned}$$

.

2. Solve

$$\begin{aligned} -x + y &= 4 \\ x^2 + y &= 3 \end{aligned}$$

Two equations, two variables

Method of elimination

1. Solve

$$\begin{aligned} 5x + 3y &= 9 \\ 2x - 4y &= 14 \end{aligned}$$

2. Solve

$$\begin{aligned} x - 2y &= 3 \\ -2x + 4y &= 1 \end{aligned}$$

3. Solve

$$\begin{aligned} 2x - y &= 1 \\ 4x - 2y &= 2 \end{aligned}$$

Analytic geometry

Study of geometry using a coordinate system.

Vectors

Vector: geometric object with *direction* and *magnitude*.

Vectors

Vector: geometric object with *direction* and *magnitude*.

Example

Suppose we are in the Euclidean plane $\mathbb{R}^2$. Consider points $p = (4, -7)$ and $q = (-1, 5)$. Draw the vector from p to q.

Vectors

Vector: geometric object with *direction* and *magnitude*.

Example

Suppose we are in the Euclidean plane $\mathbb{R}^2$. Consider points $p = (4, -7)$ and $q = (-1, 5)$. Draw the vector from p to q.

Example

Consider the vector $\vec{pq}$ from the previous example. What is its *angle*?

What can we do with vectors?

Suppose we have vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, and real numbers $\alpha, \beta \in \mathbb{R}$.

What can we do with vectors?

Suppose we have vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, and real numbers $\alpha, \beta \in \mathbb{R}$.

- *Addition*: $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2 + \cdots + u_n + v_n)$.

What can we do with vectors?

Suppose we have vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, and real numbers $\alpha, \beta \in \mathbb{R}$.

- *Addition*: $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2 + \cdots + u_n + v_n)$.
- *Scalar multiplication*: $\alpha \vec{u} = (\alpha u_1, \alpha u_2, \ldots, \alpha u_n)$.

What can we do with vectors?

Suppose we have vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, and real numbers $\alpha, \beta \in \mathbb{R}$.

- *Addition*: $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2 + \cdots + u_n + v_n)$.
- *Scalar multiplication*: $\alpha \vec{u} = (\alpha u_1, \alpha u_2, \ldots, \alpha u_n)$.

Properties of above operations:

- *Commutativity*: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

What can we do with vectors?

Suppose we have vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, and real numbers $\alpha, \beta \in \mathbb{R}$.

- *Addition*: $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2 + \cdots + u_n + v_n)$.
- *Scalar multiplication*: $\alpha \vec{u} = (\alpha u_1, \alpha u_2, \ldots, \alpha u_n)$.

Properties of above operations:

- *Commutativity*: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
- *Associativity*: $(\vec{u} + \vec{v}) + \vec{w} = \vec{v} + (\vec{u} + \vec{w})$.

What can we do with vectors?

Suppose we have vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, and real numbers $\alpha, \beta \in \mathbb{R}$.

- *Addition*: $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2 + \cdots + u_n + v_n)$.
- *Scalar multiplication*: $\alpha\vec{u} = (\alpha u_1, \alpha u_2, \ldots, \alpha u_n)$.

Properties of above operations:

- *Commutativity*: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
- *Associativity*: $(\vec{u} + \vec{v}) + \vec{w} = \vec{v} + (\vec{u} + \vec{w})$.
- *Distributivity over scalar multiplication*: $(\alpha + \beta)\vec{u} = \alpha\vec{u} + \beta\vec{u}$.

What can we do with vectors?

Suppose we have vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, and real numbers $\alpha, \beta \in \mathbb{R}$.

- *Addition*: $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2 + \cdots + u_n + v_n)$.
- *Scalar multiplication*: $\alpha\vec{u} = (\alpha u_1, \alpha u_2, \ldots, \alpha u_n)$.

Properties of above operations:

- *Commutativity*: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
- *Associativity*: $(\vec{u} + \vec{v}) + \vec{w} = \vec{v} + (\vec{u} + \vec{w})$.
- *Distributivity over scalar multiplication*: $(\alpha + \beta)\vec{u} = \alpha\vec{u} + \beta\vec{u}$.
- *Distributivity over addition*: $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$.

Length of a vector

Computing the length

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}.$$

Length of a vector

Computing the length

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}.$$

Is it true that $\|\alpha\vec{u}\| = \alpha\|\vec{u}\|$?

Length of a vector

Computing the length

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}.$$

Is it true that $\|\alpha\vec{u}\| = \alpha\|\vec{u}\|$? No, but $\|\alpha\vec{u}\| = |\alpha|\|\vec{u}\|$.

Length of a vector

Computing the length

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}.$$

Is it true that $\|\alpha\vec{u}\| = \alpha\|\vec{u}\|$? No, but $\|\alpha\vec{u}\| = |\alpha|\|\vec{u}\|$.

Computing the unit vector

$$\frac{\vec{u}}{\|\vec{u}\|}.$$

An airplane is descending at 200 km/hr at an angle of 30 degrees below the horizon. Find the component form of its velocity vector.

Dot product

Definition

Suppose we have $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$. The *dot product*[1] *of* $\vec{u}$ *and* $\vec{v}$ is defined as

$$\vec{u} \cdot \vec{v} = (u_1 v_1, u_2 v_2, \ldots, u_n v_n).$$

[1] You will see during your studies that there are multiple types of dot products. This one is usually known as the *standard dot product*.

Dot product

Definition

Suppose we have $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$. The *dot product*[1] *of* $\vec{u}$ *and* $\vec{v}$ is defined as

$$\vec{u} \cdot \vec{v} = (u_1 v_1, u_2 v_2, \ldots, u_n v_n).$$

Properties

- *Commutativity*: $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.

[1] You will see during your studies that there are multiple types of dot products. This one is usually known as the *standard dot product*.

Dot product

Definition

Suppose we have $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$. The *dot product*[1] *of* $\vec{u}$ *and* $\vec{v}$ is defined as

$$\vec{u} \cdot \vec{v} = (u_1 v_1, u_2 v_2, \ldots, u_n v_n).$$

Properties

- *Commutativity*: $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.
- $\vec{0} \cdot \vec{v} = \vec{0}$.

[1] You will see during your studies that there are multiple types of dot products. This one is usually known as the *standard dot product*.

Dot product

Definition

Suppose we have $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$. The *dot product*[1] *of* $\vec{u}$ *and* $\vec{v}$ is defined as

$$\vec{u} \cdot \vec{v} = (u_1 v_1, u_2 v_2, \ldots, u_n v_n).$$

Properties

- *Commutativity*: $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.
- $\vec{0} \cdot \vec{v} = \vec{0}$.
- *Distributivity*: $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.

[1] You will see during your studies that there are multiple types of dot products. This one is usually known as the *standard dot product*.

Dot product

Definition

Suppose we have $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$. The *dot product*[1] *of* $\vec{u}$ *and* $\vec{v}$ is defined as

$$\vec{u} \cdot \vec{v} = (u_1 v_1, u_2 v_2, \ldots, u_n v_n).$$

Properties

- *Commutativity*: $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.
- $\vec{0} \cdot \vec{v} = \vec{0}$.
- *Distributivity*: $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.
- $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$.

---

[1] You will see during your studies that there are multiple types of dot products. This one is usually known as the *standard dot product*.

Dot product

Definition

Suppose we have $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$. The *dot product*[1] *of* $\vec{u}$ *and* $\vec{v}$ is defined as

$$\vec{u} \cdot \vec{v} = (u_1 v_1, u_2 v_2, \ldots, u_n v_n).$$

Properties

- *Commutativity*: $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.
- $\vec{0} \cdot \vec{v} = \vec{0}$.
- *Distributivity*: $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.
- $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$.
- *Triangle inequality*: $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$.

[1] You will see during your studies that there are multiple types of dot products. This one is usually known as the *standard dot product*.

Dot product in the plane

Let $\vec{u}, \vec{v} \in \mathbb{R}^2$, and θ be the angle between $\vec{u}$ and $\vec{v}$. Then

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta.$$

Dot product in the plane

Let $\vec{u}, \vec{v} \in \mathbb{R}^2$, and θ be the angle between $\vec{u}$ and $\vec{v}$. Then

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta.$$

θ in degrees	θ in radians	$\vec{u} \cdot \vec{v}$
90°	$\frac{\pi}{2}$ rad	0
0°	0 rad	$\|\vec{u}\| \|\vec{v}\|$
180°	π rad	$-\|\vec{u}\| \|\vec{v}\|$

Projection

Definition

Projection of vector $\vec{u}$ on vector v is the vector

$$\operatorname{proj}_v(u) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|} \cdot \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \cdot \vec{v}.$$

Circles

Definition

A *circle* is a set of equidistant points from a fixed point (h, k) called the *center*. The distance from the center to any of the circle points is called the *radius*.

Circles

Definition

A *circle* is a set of equidistant points from a fixed point (h, k) called the *center*. The distance from the center to any of the circle points is called the *radius*.

Standard form of the equation of a circle

$$(x - h)^2 + (y - k)^2 = r^2$$

1. A circle has center $(2, 3)$ and includes the point $(1, 4)$. Find its standard equation.

1. A circle has center $(2, 3)$ and includes the point $(1, 4)$. Find its standard equation.
2. Find the center and the radius of a circle

$$x^2 - 6x + y^2 - 2y + 6 = 0.$$

Ellipses

Definition

An *ellipse* is the set of points whose sum of distances from two distinct points called foci is constant.

Ellipses

Definition

An *ellipse* is the set of points whose sum of distances from two distinct points called foci is constant.

Some terminology:

- *Center* is the midpoint of the foci.

Ellipses

Definition

An *ellipse* is the set of points whose sum of distances from two distinct points called foci is constant.

Some terminology:

- *Center* is the midpoint of the foci.
- *Major axis* is the chord through the foci.

Ellipses

Definition

An *ellipse* is the set of points whose sum of distances from two distinct points called foci is constant.

Some terminology:

- *Center* is the midpoint of the foci.
- *Major axis* is the chord through the foci.
- The major axis intersects the ellipse at *vertices*.

Ellipses

Definition

An *ellipse* is the set of points whose sum of distances from two distinct points called foci is constant.

Some terminology:

- *Center* is the midpoint of the foci.
- *Major axis* is the chord through the foci.
- The major axis intersects the ellipse at *vertices*.
- *Minor axis* is the chord through the center perpendicular to the major axis.

Ellipses

Definition

An *ellipse* is the set of points whose sum of distances from two distinct points called foci is constant.

Some terminology:

- *Center* is the midpoint of the foci.
- *Major axis* is the chord through the foci.
- The major axis intersects the ellipse at *vertices*.
- *Minor axis* is the chord through the center perpendicular to the major axis.
- The minor axis intersects the ellipse at *co-vertices*.

Properties of ellipses

- Consider an ellipse with center at (h, k), foci at $(h \pm c, k)$, vertices at $(h \pm a, k)$, and co-vertices at $(h, k \pm b)$.

Properties of ellipses

- Consider an ellipse with center at (h, k), foci at $(h \pm c, k)$, vertices at $(h \pm a, k)$, and co-vertices at $(h, k \pm b)$.
- Sum of distance to foci is $(a + c) + (a - c) = 2a$.

Properties of ellipses

- Consider an ellipse with center at (h, k), foci at $(h \pm c, k)$, vertices at $(h \pm a, k)$, and co-vertices at $(h, k \pm b)$.
- Sum of distance to foci is $(a + c) + (a - c) = 2a$.

$\Rightarrow$ Distance from a focal point to a co-vertex is a.

Properties of ellipses

- Consider an ellipse with center at (h, k), foci at $(h \pm c, k)$, vertices at $(h \pm a, k)$, and co-vertices at $(h, k \pm b)$.
- Sum of distance to foci is $(a + c) + (a - c) = 2a$.

$\Rightarrow$ Distance from a focal point to a co-vertex is a.

$\Rightarrow$ $c^2 = a^2 - b^2$.

Properties of ellipses

- Consider an ellipse with center at (h, k), foci at $(h \pm c, k)$, vertices at $(h \pm a, k)$, and co-vertices at $(h, k \pm b)$.
- Sum of distance to foci is $(a + c) + (a - c) = 2a$.

$\Rightarrow$ Distance from a focal point to a co-vertex is a.

$\Rightarrow$ $c^2 = a^2 - b^2$.

- *Eccentricity* of an ellipse is defined as $\frac{c}{a}$.

Properties of ellipses

- Consider an ellipse with center at (h, k), foci at $(h \pm c, k)$, vertices at $(h \pm a, k)$, and co-vertices at $(h, k \pm b)$.
- Sum of distance to foci is $(a + c) + (a - c) = 2a$.

$\Rightarrow$ Distance from a focal point to a co-vertex is a.

$\Rightarrow$ $c^2 = a^2 - b^2$.

- *Eccentricity* of an ellipse is defined as $\frac{c}{a}$.

Standard equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

1. Find the equation of an ellipse with foci at $(0, 1)$ and $(4, 1)$ and major axis of length 6.

1. Find the equation of an ellipse with foci at $(0, 1)$ and $(4, 1)$ and major axis of length 6.
2. Find the center and vertices of an ellipse $x^2 + 4y^2 + 6x - 8y + 9 = 0$.

Cross product

Only defined in three dimensional spaces.

Definition

The *cross product* of $\vec{u}, \vec{v} \in \mathbb{R}^3$ is defined as

$$\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1).$$

Cross product: geometric properties

Let $\vec{u}, \vec{v} \in \mathbb{R}^3$, and θ be the angle between them.

Cross product: geometric properties

Let $\vec{u}, \vec{v} \in \mathbb{R}^3$, and θ be the angle between them.

- $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.
  - The "orthogonal direction" is determined by convention.

Cross product: geometric properties

Let $\vec{u}, \vec{v} \in \mathbb{R}^3$, and θ be the angle between them.

- $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.
  - The "orthogonal direction" is determined by convention.
- $\vec{u} \times \vec{v} = \|\vec{u}\|\|\vec{v}\| \sin(\theta)\vec{n}$ where $\vec{n}$ is the unit vector orthogonal to $\vec{u}$ and $\vec{v}$.

Cross product: geometric properties

Let $\vec{u}, \vec{v} \in \mathbb{R}^3$, and θ be the angle between them.

- $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.
  - The "orthogonal direction" is determined by convention.
- $\vec{u} \times \vec{v} = \|\vec{u}\|\|\vec{v}\| \sin(\theta)\vec{n}$ where $\vec{n}$ is the unit vector orthogonal to $\vec{u}$ and $\vec{v}$.
- $\|\vec{u} \times \vec{v}\|$ is the area of the parallelogram between $\vec{u}$ and $\vec{v}$.

Lines and planes

Parametric equation of a line

Let $t \in \mathbb{R}$ be a parameter.

$$x = x_1 + at; y = y_1 + bt; z = z_1 + bt$$

Symmetric equation of a line

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

Lines and planes

Parametric equation of a line

Let $t \in \mathbb{R}$ be a parameter.

$$x = x_1 + at; y = y_1 + bt; z = z_1 + bt$$

Symmetric equation of a line

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

Exercise

Find the parametric and the symmetric equation of a line passing through points $(-2, 1, 0)$ and $(1, 3, 5)$.

Lines and planes

- Consider a plane that passes through the point (x_1, y_1, z_1) and has a normal vector (a, b, c).

Lines and planes

- Consider a plane that passes through the point (x_1, y_1, z_1) and has a normal vector (a, b, c).
- Then for any point (x, y, z) in the plane we have

$$(a, b, c) \cdot (x - x_1, y - y_1, z - z_1) = 0.$$

Lines and planes

- Consider a plane that passes through the point (x_1, y_1, z_1) and has a normal vector (a, b, c).
- Then for any point (x, y, z) in the plane we have

$$(a, b, c) \cdot (x - x_1, y - y_1, z - z_1) = 0.$$

$\Rightarrow$ Standard equation of a plane

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

Lines and planes

- Consider a plane that passes through the point (x_1, y_1, z_1) and has a normal vector (a, b, c).
- Then for any point (x, y, z) in the plane we have

$$(a, b, c) \cdot (x - x_1, y - y_1, z - z_1) = 0.$$

$\Rightarrow$ Standard equation of a plane

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

- General form of the equation of a plane

$$ax + by + cz + d = 0.$$

1. Find the general equation of the plane passing through $(2, 1, 1)$, $(0, 4, 1)$, and $(-2, 1, 4)$.

1. Find the general equation of the plane passing through $(2, 1, 1)$, $(0, 4, 1)$, and $(-2, 1, 4)$.
2. Find the intersection of planes $x - 2y + z = 0$ and $2x + 3y - 2z = 0$.