1. I have drawn a number X from Geom(1/2) and I am hiding its value from you. The "Geom(1/2)" part means that $\Pr[X = k] = 2^{-k}$ for any $k \in \mathbb{N}$. You want to find the value of X. I will truthfully answer any yes/no question you give me of the form "Is X contained in set S?". Find a strategy that minimizes the expected number of questions you need to ask until you find the value of X. Compare the required expected number of questions with $H(X)$.

2. Let X be a finite random variable. What is the (general) inequality relationship between $H(X)$ and $H(Y)$ if:

   a) $Y = 2^X$,

   b) $Y = \cos(X)$.

3. Let X be a random variable and g a function of X. Show that $H(g(X)) \leq H(X)$ by justifying the following steps.

   1. $H(X, g(X)) = H(X) + H(g(X)|X) = H(X)$.
   2. $H(X, g(X)) = H(g(X)) + H(X|g(X)) \geq H(g(X))$.
   3. Thus $H(X) \geq H(g(X))$.

4. Imagine that you are taking the exam of this course and the question you get is "Prove that relative entropy[1] is always non-negative". But you do not want to use Jensen's inequality because you forgot its proof. Fortunately, as a good algorithm designer, you know how to prove $1 + x \leq e^x$ for every $x \in \mathbb{R}$ and derive $\ln(z) \leq z - 1$.[2] How can you use this to solve the exam question?

5. You have an urn with r red balls and g green balls. You want to randomly sample b balls. You can either do it with replacement[3] or without replacement[4]. Which distribution has larger entropy – with replacement or without?

6. Imagine that you have a bent coin – it lands on heads (H) with probability $0 < p < 0.5$ and on tails (T) with probability $1 - p$. You want to simulate a fair coin. A classic strategy is to flip $2n$ times to get $X_1, \ldots, X_{2n}$, for $k \in [n]$ if $X_{2k-1}X_{2k}$ is HT, you interpret it as H, if it is TH, you interpret it as T, and you ignore the remaining outcomes (do you see that we get H's or T's uniformly independently?). How many fair coin flips does this strategy produce in expectation? What is the maximal number of coin flips that a strategy can produce in expectation?

   To add formalism, suppose that we have an arbitrary mapping that takes a sequence $X_1, \ldots, X_n$ of flips of a bent coin and produces a sequence of fair coin flips $Z_1, \ldots, Z_K$ with the property that $Z_1, \ldots, Z_K$ is uniformly distributed over fair heads-tails sequences of length K. Note that K may depend on $X_1, \ldots, X_n$. Prove that $nH(p) \geq \mathbb{E}[K]$.

   Note. It might seem that the task is solved by $H(X) \geq H(g(X))$ but you also need to go from $H(g(X))$ to $\mathbb{E}[K]$ somehow.

7. The goal of this task is to see that entropy can be infinite.

   Let $A = \sum_{n=2}^{\infty} (n \log^2 n)^{-1}$. You can trust me that A converges (to 2.10874). Show that a random variable with distribution $\Pr[X = n] = (An \log^2 n)^{-1}$ for $n = 2, 3, \ldots$ has $H(X) = \infty$.

[1] Also known as Kullbeck-Leibler divergence.
[2] The examiner might not even ask for the proof because it is easy.
[3] Meaning that after removing a ball and inspecting its color, you return it to the urn.
[4] Meaning that after removing a ball and inspecting its color, you destroy it.