Probability Review

February 25, 2025

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Exercise

Let $\mathcal{S} \subseteq 2^{\{1,2,\ldots,n\}}$ (i.e. subsets of $\{1,2,\ldots,n\}$). A 2-coloring of $\{1,2,\ldots,n\}$ is any function $f\colon \{1,2,\ldots,n\} \to \{\mathrm{red}, \mathrm{blue}\}$.
Show that if $\forall S \in \mathcal{S}$:

- $\forall S \in \mathcal{S}\colon |S| = k$ (assume $n > k$), and
- $|\mathcal{S}| < 2^{k-1}$,

then there exists a 2-coloring of $\{1,2,\ldots,n\}$ such that no $S \in \mathcal{S}$ is monochromatic.

Conditional probability

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Exercise

Alice and Bob are throwing darts at a target. Alice has a chance $1/3$ of hitting the target, and Bob has chance $1/4$ of doing the same. What is the chance Alice hits the target given that at least one of them hit it?

Events $A_1, \ldots, A_m$ are mutually independent if for all $I \subseteq \{1, 2, \ldots, m\}$:

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Chain rule

$$\Pr[\cap_{i=1}^{m} A_i] = \Pi_{i=1}^{m} \Pr[A_i | A_{i-1} \cap A_{i-2} \cap \cdots \cap A_1].$$

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- A random variable X is independent of event A if $\forall S \subseteq \mathbb{R} : \Pr[X \in S | A] = \Pr[X \in S]$.
- Random variables X, Y are independent if $\forall S, T \subseteq \mathbb{R}$, "$X \in S$" and "$Y \in T$" are independent events.

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- For an integer random variable $X \colon \Omega \to \mathbb{N}_0$, we have $\mathbb{E}[X] = \sum_{k=0}^{\infty} \Pr[X \geq k]$.

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Suppose that we are looking with our perfect vision at the Pentagon at from an angle which is uniformly random. What is the expected number of sides of the Pentagon that we see?

Markov's and Chebyshev's inequality

Markov's inequality

Let $X\colon \Omega \to \mathbb{R}^+$ be a positive random variable. Then for any $t > 0$ we have

$$\Pr[X \geq t] \leq \mathbb{E}[X]/t.$$

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Chebyshev's inequality

Let X be a random variable. For any $t \in \mathbb{R}^+$

$$\Pr[|X - \mathbb{E}[X]| \geq t] \leq \mathrm{Var}[X]/t^2.$$

Chernoff bounds

Let $X_1, \ldots, X_n$ be independent 0/1 random variables satisfying $\Pr[X_i = 1] = p$ for some $p \leq 1/2$. Let $X = \sum_{i=1}^{n} X_i$ and $\mu = \mathbb{E}[X] = np$.

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Proof

On the board.