Probability review

- probability space Ω: finite (or at most countable for us) set w/ a measure
  $Pr : \Omega \to \mathbb{R}^+$ satisfying
  - $\forall \omega \in \Omega : Pr(\omega) \geq 0$
  - $\sum_{\omega \in \Omega} Pr(\omega) = 1$

- event $A \subseteq \Omega$ has $Pr[A] = \sum_{\omega \in \Omega} Pr[\omega]$
  - $Pr[A] = 1 - Pr[\Omega \setminus A]$
- union bound: Let $A_1, \dots, A_n$ be events. Then
  $Pr[\bigcup_{i=1}^{n} A_i] \leq \sum_{i=1}^{n} Pr[A_i]$

Exercise: Let $\mathcal{G} \subseteq [n] = \{1, 2, \dots, n\}$. A 2-coloring of $[n]$ is any fct $f : [n] \to \{1,2\}$.
Show that if if $\forall S \in \mathcal{G} : |S| = k$ and $|\mathcal{G}| \leq 2^{k-1}$ (assume $n > k$), then there exists a 2-coloring of $[n]$ s.t. no $S \in \mathcal{G}$ is monochromatic.

Conditional probability

- the *conditional probability of event A on event B* $\Pr[A|B]$ is

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]}$$

- event A is *independent of event B* if $\Pr[A|B] = \Pr[A]$
  - implies $\Pr[A \cap B] = \Pr[A]\Pr[B]$

Exercise: Alice and Bob are throwing darts at a target. Alice has a chance $\frac{1}{3}$ of hitting the target, Bob $\frac{1}{4}$. What is the chance Alice hit the target given that at least one of them hit it?

Mutual independence

- events $A_1, A_2, \dots, A_n$ are mutually independent if for all $I \subseteq [n]$

$$\Pr\left[\bigcap_{i \in I} A_i\right] = \prod_{i \in I} \Pr[A_i]$$

Chain rule

- $\Pr[A \cap B \cap C] = \Pr[A]\Pr[B \mid A]\Pr[C \mid A \cap B] \dots$
- in general $\Pr\left[\bigcap_{i=1}^{n} A_i\right] = \prod_{i=1}^{n} \Pr[A_i \mid A_{i-1} \cap A_{i-2} \cap \dots A_1]$

Random variables

- a random variable X is a measurable fct $X : \Omega \to \mathbb{R}$

Example: for a $t \in \mathbb{R}$, "$X \geq t$" is an event $\{\omega \in \Omega : X(\omega) \geq t\}$

- a random variable X is indep of event A if $\forall S \subseteq \mathbb{R}$ $\Pr[X \in S \mid A] = \Pr[X \in S]$
- r.v.'s X, Y are indep if $\forall S, T \subseteq \mathbb{R}$, $X \in S$ and $Y \in T$ are indep events

Expectation of r.v.'s

- *expectation of an r.v. X* is $\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \Pr[\omega]$

Linearity of expectation

- for random variables (not necessarily indep!!!) $X_1, \ldots, X_n$ and coefficients $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, we have $\mathbb{E}[\lambda_1 X_1 + \lambda_2 X_2 + \ldots + \lambda_n X_n] = \sum_{i=1}^{n} \lambda_i \mathbb{E}[X_i]$

- if X, Y are indep r.v.'s, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$

- for any const $t \in \mathbb{R}$, $\mathbb{E}[t] = t$

- for an integer r.v. $X : \Omega \to \mathbb{N}$, we have $\mathbb{E}[X] = \sum_{k=1}^{\infty} \Pr[X \geq k]$

Exercise: Suppose we keep tossing a biased coin where a heads lands with pr. p and tails w/ pr. $1-p$. What is the expected # tosses until we get first heads?

Exercise: Shuffle a deck of cards and start revealing cards one by one. What is the expected # cards we reveal until we get the first ace?

Exercise: Suppose we are standing very very very far away from the Pentagon and we have perfect vision. What is the expected number of sides that we see?

Markov inequality: Let $X : \Omega \to \mathbb{R}^+$ be a positive r.v. Then for any $t > 0$ we have

$$\Pr[X \geq t] \leq \frac{\mathbb{E}[X]}{t}$$

Variance

· Variance of a r.v. X is $Var[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$.

· $\mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2 - 2X\mathbb{E}[X] + \mathbb{E}[X]^2] = \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + \mathbb{E}[X]^2 =$
$= \mathbb{E}[X^2] - \mathbb{E}[X]^2$

· for a constant $c \in \mathbb{R}$, $Var[cX] = c^2 Var[X]$

· for mutually indep r.v.'s $X_1, \dots, X_n$, $Var[X_1 + X_2 + \dots + X_n] = \sum_{i=1}^{n} Var[X_i]$

Chebyshev inequality: Let X be an r.v. For any $t \in \mathbb{R}^+$, $\Pr[|X - \mathbb{E}[X]| > t] \leq \frac{Var[X]}{t^2}$

<u>Chernoff bounds</u>: Let $X_1, \ldots, X_n$ be <u>indep</u>!!! 0-1 r.v.'s. Let $p_i = \Pr[X_i = 1]$, thus $1 - p_i = \Pr[X_i = 0]$. Let $X = \sum_{i=1}^{n} X_i$, $\mu = \mathbb{E}[X] = \sum_{i=1}^{n} p_i$. For any $\delta \in (0,1)$

$$\Pr[X \geq (1+\delta)\mu] \leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu} \qquad \left(\approx 2^{-c \cdot \delta \cdot n} \text{ for some } c \in \mathbb{R}\right)$$

and

$$\Pr[X \leq (1-\delta)\mu] \leq e^{-\frac{1}{2}\mu\delta^2}$$

<u>Exercise</u>: Design <u>dependent</u>! r.v.'s X_i where Chernoff bound does not hold