Definition. Let $\mathcal{A}$ be a family of sets (without duplicates) over a universe $\mathcal{U}$. A *sunflower* with k *petals* and a core Y is a collection of sets $S_1, \dots, S_k \in \mathcal{A}$ such that $S_i \cap S_j = Y$ for all $i \neq j$; the sets $S_i \setminus Y$ are *petals* and we require none of them to be empty. Note that a family of pairwise disjoint sets is a sunflower (with an empty core).

Sunflower lemma. Let $\mathcal{A}$ be a family of sets (without duplicates) over a universe $\mathcal{U}$, such that each set in $\mathcal{A}$ has cardinality exactly d. If $|\mathcal{A}| > d!(k-1)d$, then $\mathcal{A}$ contains a sunflower with k petals and such a sunflower can be computed in time polynomial in $|\mathcal{A}|$, $|\mathcal{U}|$, k.

Problem: d-Hitting Set
Input: A family of sets $\mathcal{A}$ (without duplicates) over a universe $\mathcal{U}$, such that each set in $\mathcal{A}$ has cardinality exactly d, integer $k \in \mathbb{N}$.
Output: Is the a subset $H \subseteq \mathcal{U}$ of size at most k such that every set $S \in \mathcal{A}$ contains at least one element of H?

1. Design a kernel for d-Hitting Set with at most $d!k^d d$ sets and at most $d!k^d d^2 elements$.

2. Have you ever wondered how to prove that a problem with an FPT algorithm does not have a polynomial kernel? In this exercise I will sketch something like that. More details are in Section 15 of the book.

   In the Longest Path problem, we are given a graph $G = (V, E)$, and an integer k. The goal is to find a path of length at least k (meaning that it has to visit k different vertices).

   Suppose there exists a kenel for Longest Path of size k^3. Now, let us take k^7 instances of Longest Path and create a new instance I from their disjoint union. Clearly, I is a yes-instance if and only if one of the original instances was a yes-instance.

   Once we apply the kernelization procedure on I, something has to be wrong with the resulting "kernel". Can you find what it is?

   This result is conditional: in literature you will see statements like "Problem X does not admit a polynomial kernel unless $\mathsf{NP} \subseteq \mathsf{coNP/poly}$".

3. We will now make a guided proof of Sunflower lemma.

   We prove it by induction on d. The statement is trivial for $d = 1$ as a family of pairwise disjoint sets form a sunflower.

   So suppose $d \geq 2$, let $\mathcal{A}$ be a family of sets of cardinality d over universe $\mathcal{U}$ such that $|\mathcal{A}| > d!(k-1)^d$. Let $\mathcal{G} = \{S_1, \dots, S_\ell\} \subseteq \mathcal{A}$ be an inclusion-wise maximal family of pairwise disjoint sets in $\mathcal{A}$.

   - If $\ell \geq k$, then we are done. Why?

   So suppose $\ell < k$. Let $S = \bigcup_{i=1}^{\ell} S_i$. Because $\mathcal{G}$ is maximal, every set $A \in \mathcal{A}$ intersects at least one set from $\mathcal{G}$, i.e., $A \cap S \neq \emptyset$.

   - Using this fact, show that there is an element $u \in \mathcal{U}$ contained in many sets from $\mathcal{A}$ (maybe by using the averaging argument; then find a upper bound on $|S|$). We will remove this element u, apply the induction hypothesis on sets which used to contain u to obtain a sunflower and add it back to get a sunflower. So make sure that "many" is large enough.
   - And use the proof to give a polynomial algorithm for finding a sunflower.

4. Show that Feedback Vertex Set parameterized by solution size k on undirected d-regular graphs admits a kernel of size $\mathcal{O}(k)$.

5. An undirected graph G is called *perfect* if for every induced subgraph H of G, the size of the largest clique in H is same as the chromatic number of H. Obtain a kernel with $\mathcal{O}(k^2)$ vertices for Odd Cycle Transversal on perfect graphs.

   Hint: Give a necessary and sufficient condition for a perfect graph being bipartite.

No homework this week :)