Note that any algorithm that is given a graph along with its tree decomposition, and runs in FPT time, can be turned into an FPT algorithm without a given decomposition:
Theorem (Approximate tree decomposition in FPT time): There is an algorithm that, given an n-vertex graph G and integer k, in time $\mathcal{O}(8^k k^2 \cdot n^2)$ either concludes that $tw(G) > k$, or constructs a tree decomposition of G of width $\leq 4k + 4$.

Monadic second-order logic on graphs

Formulas of $\mathbf{MSO}_2$ can use four types of variables:

1. single vertices,
2. single edges,
3. subsets of vertices,
4. subsets of edges.

(The subscript 2 in $\boldsymbol{MSO}_2$ signifies that quantification over subsets of edges is allowed ($\boldsymbol{MSO}_1$ only allows the first three types of variables). This style of notation is inherited from model theory.)

If we denote by Σ the language of $\mathbf{MSO}_2$ logic, a model of a formula φ is a structure $\langle G, \Sigma^G \rangle$. Σ^G gives evaluations to variables in φ.
These are the allowed atomic formulas and their semantics:

- $x \in X$ holds when vertex (edge) x^G belongs to vertex (edge) subset X^G,
- $\mathbf{inc}(v, e)$ holds when vertex v^G is an endpoint of edge e^G,
- $x = y$ is a valid atom for x and y of the same type (and holds in $\langle G, \Sigma^G \rangle$ when $x^G = y^G$).

Example: subgraph connectivity

$$\mathbf{conn}(X) = \forall Y \subseteq V \, [(\exists u, v \in X : u \in Y \wedge v \notin Y) \\ \Rightarrow (\exists e \in E \; \exists u, v \in X : \mathbf{inc}(u, e) \wedge \mathbf{inc}(v, e) \wedge u \in Y \wedge v \notin Y)] \, .$$

For a formula φ by $\|\varphi\|$ we denote the length of the encoding of φ as a string.

Theorem (Courcelle): There is an algorithm that, given

- an $\mathbf{MSO}_2$ formula φ,
- an n-vertex graph G equipped with an evaluation of all the free variables of φ, and
- a tree decomposition of G of width t,

verifies whether φ is satisfied in G in time $f(\|\varphi\|, t) \cdot n$, for a computable f.

Exercise: Express the 3-colorability of a graph as an $\mathbf{MSO}_2$ formula.

Exercise: Write an $\mathbf{MSO}_2$ formula that is satisfied for graphs with a dominating set of size at most k.

Theorem (Courcelle, optimization version): Let $\alpha(x_1, x_2, \ldots, x_p)$ be an affine function. There is an algorithm that, given

- an $\mathbf{MSO}_2$ formula φ with p free monadic variables $X_1, \ldots, X_p$,
- an n-vertex graph G equipped with an evaluation of all the free variables of φ apart from $X_1, \ldots, X_p$, and
- a tree decomposition of G of width t,

in time $f(\|\varphi\|, t) \cdot n$ finds the minimum and maximum value of $\alpha(|X_1|, \ldots, |X_p|)$ over sets $X_1, \ldots, X_p$ satisfying $\varphi(X_1, \ldots, X_p)$, where f is a computable function.

Exercise: Use the optimization version to obtain an FPT algorithm for dominating set parameterized by treewidth.

Exercise: Show how, given a graph H, you would construct an $\mathbf{MSO}_2$ formula valid for graphs that contain H as a minor.

Exercise: Write an $\mathbf{MSO}_2$ formula that is valid for hamiltonian graphs.

Exercise: Use the optimization version of Courcelle's theorem to solve the *Minimum Feedback Vertex Set* problem in FPT parameterized by treewidth.