TREEWIDTH & MSO_2

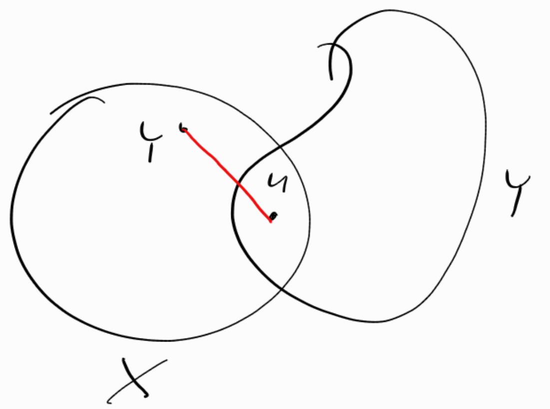
- See 7.4 in the book

- many problems are FPT in tw
- usually via DP
- DP ... annoying and all the algs are "the same"
- MSO_2: unified (programming) language to specify problems

Example: Is $X \subseteq V(G)$ a connected subgraph of G?

$$\mathrm{conn}(X) = \forall Y \subseteq V \; (\exists u \in X \; u \in Y \land \exists v \in X \; v \in Y)$$

$$\Rightarrow (\exists e \in E \; \exists u \in X \; \exists v \in X \; \underline{\mathrm{inc}}(u,e) \land \mathrm{inc}(v,e) \land u \in Y \land v \notin Y)$$



- "For every subset of vertices Y, if X contains both a vertex from Y and a vertex outside of Y, then there exists an edge e whose endpoints u,v belong to X but one endpoint belongs to Y and the other is outside of Y.

$\Leftrightarrow$ this encodes connectivity of X: there is no partitioning of V(G) into Y & V(G) s.t. X is partitioned non trivially & no edge of G[X] crosses the partition

$\Leftrightarrow$ each cut as a vertex subset and its complement has ≥ 1 edge

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MSO_2 as a prog lang

variables
- for vertices, edges, vertex subsets, edge subsets

functions
- we can define a fla that has parameters & the fla can eval to true iff the passed args satisfy the fla (are a model?)

operators
- standard Bool. connectives $(\wedge, \vee, \Rightarrow, \neg, \dots)$
- $inc(u,e) = \begin{cases} \text{true} & \text{vertex } u \text{ is incident to edge } e \\ \text{false} & \text{o/w} \end{cases}$
- equality

loops
- $\exists$ and $\forall$ quantifiers over vertex or edge subsets
- e.g. $\exists u \in V, \forall e \in E, \exists X \subseteq V, \exists C \subseteq E(G), \dots$

- we occasionally use hacks
  - $A \subseteq B : \forall v \in V(G) : v \in A \Rightarrow v \in B$
  - $u \neq v : \neg(u = v)$
  - $adj(u,v) : (u \neq v) \wedge (\exists e \in E \; inc(u,e) \wedge inc(v,e))$

  $\vdots$

Courcelle's theorem
- if f is a fla, then $\|\varphi\|$ is the length of encoding of φ as a string

Thm:

- φ ... MSO_2 fla
- G ... gr
- verifying if G satisfies φ can be done in time $f(\|\varphi\|, tw(G))\, n$

  for some computable fct f

- from my memory $f = 2^{2^{2^{\cdot^{\cdot^{\cdot^{2^{\|\varphi\|}}}}}}} \Big\} tw$

$\to$ takeaway: want a FPT alg in tw? Write a "short" MSO_2 fla.

Ex: vertex cover of size k

- $\exists x_1, \dots, x_k \in V : \forall e \in E \;\; inc(x_1, e) \vee inc(x_2, e) \vee \dots \vee inc(x_k, e)$

- "there exist k vertices $x_1, x_2, \dots, x_k$ such that each edge is inc to at least

  one of $x_1, \dots, x_k$"

$\to$ $2^{2^{\cdot^{\cdot^{\cdot^{2^{O(k)}}}}}} \Big\} tw$ alg

- but we know that there exists an algo for VC on bounded tw grs

  that is poly in sol size (i.e. k does not appear in the superpoly part)

- is there a way to do this?

Thm: (Courcelle's thm: Optimization version):

- let φ be a fla w/ monadic vars $X_1, \dots, X_p$ (i.e. vertex or edge subsets)
- $\alpha : (V \cup E)^p \to \mathbb{R}$

- then there exists a $f(\|\varphi\|, tw) \cdot n$ algo that finds a setting of $X_1, \dots, X_p$ that satisfies $\varphi(X_1, \dots, X_p)$ (if one exists) and minimizes/maximizes $\alpha(|X_1|, |X_2|, \dots, |X_p|)$

Ex: Vertex cover one more time

- $\exists X \subseteq V : \forall e \in E \ \exists u \in X : adj(u, e)$
- $\alpha(X) = |X|$
- fla has const size
- ✓

HAMILTONIAN CYCLE

$\exists F \subseteq E(G) : conn_E(F) \wedge \forall v \in V \ deg_2(v, C)$

where

$conn_E(F) : (\forall Y \subseteq V) \Big[(\exists u \in V \ u \in Y \wedge \exists v \in V \ v \notin Y)$

$\Rightarrow (\exists e \in F \ \exists u \in Y \ \exists v \notin Y \ inc(u, e) \wedge inc(v, e) \Big]$

- every cut Y is crossed by some edge e

$deg_2(v, F) : \exists e_1, e_2 \in F \Big[\big((e_1 \neq e_2) \wedge inc(v, e_1) \wedge inc(v, e_2) \wedge (\forall e_3 \in F \ inc(v, e_3)$

$\Rightarrow (e_1 = e_3) \vee (e_2 = e_3)) \big) \Big]$

- there are 2 edges such that they are different, they are both incident to v and every edge incident to v is one of these two

DOMINATING SET

- $\exists X \subseteq V \ \forall u \in V \ \exists v \in X : \ adj(u,v)$

  minimize $|X|$

FEEDBACK VERTEX SET

- $\exists X \subseteq V :$ $G \setminus X$ has no cycle (reuse "has cycle" from HAMILTONIAN CYCLE)

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