1. What is the treewidth of the following graphs? If you are struggling, try to just give an upper bound.

   a) A complete graph K_n.

   b) A complete bipartite graph $K_{n,m}$.

   c) A forest.

   d) A cycle C_n.

   e) A cube Q_3.

   f) A $m \times n$ grid $\boxplus_{m,n}$.

2. Show that for every minor H of graph G, we have $\text{tw}(H) \leq \text{tw}(G)$.

3. Prove the following lemma. It says something to the effect that given a tree decomposition, we can always find a nice tree decomposition in polynomial time.

   If a graph G admits a tree decomposition of width at most k, then it also admits a *nice tree decomposition* of width at most k. Moreover given a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ of G of width at most k, one can in time $\mathcal{O}(k^2 \cdot \max\{|V(T)|, |V(G)|\})$ compute a nice tree decomposition of G of width at most k that has at most $\mathcal{O}(k|V(G)|)$ nodes.

4. How can subdividing an edge of a graph G change its treewidth? Can in increase or decrease?

5. Show that the treewidth of a graph G is equal to the maximum treewidth of its biconnected components.

6. A graph G is *d-degenerate* if every subgraph of G contains a vertex of degree at most d.

   Prove that graphs of treewidth t are t-degenerate.

7. For a graph G given together with its tree decomposition of width t, construct in time $t^{\mathcal{O}(1)}n$ a data structure such that for any two vertices $x, y \in V(G)$ it is possible to check in time $\mathcal{O}(t)$ if x and y are adjacent.

   Now do it without results on hashing, hash tables, etc.

8. We define a *k-tree* inductively: a clique on $k+1$ vertices is a k-tree. A new k-tree G can be obtained from a smaller k-tree G' by adding a new vertex and making it adjacent to k vertices of G' that form a clique in G'. Show that every k-tree is a chordal graph of treewidth k. Prove that for every graph G and integer k, G is a subgraph of a k-tree if and only if $\text{tw}(G) \leq k$.