3k kernel FOR VC USING CROWN DECOMPOSITION

recall that there exists a poly(|G|) alg for finding a crown decomp (CD) which is a partitioning of G into three distinct vertex sets $C \dot\cup H \dot\cup R$ s.t.

1. $C \neq \emptyset$
2. C is an IS
3. there are no edges between C & R
4. let E' be $E(G[C \cup H])$. Then $|E'| = |H|$ and there exists a matching from H into C

Kernel:

rule 1: Kill deg 0 vertices

rule 2: If $|G| \leq 3k$, $k \geq 0$, done

rule 3: Find a CD $C \dot\cup H \dot\cup R$. Let M be a matching of H into C

- if $|H| \geq k+1$, we need $\geq k+1$ vertices to cover $E(M) \to$ reject
- o/w H covers all edges incident to $G[H \cup C]$
- reduce to $(G - H, k - |H|)$
- observe that rule 1 kills C

LONGEST PATH & COLOR CODING

- Longest Path
  - input: gr. G, $k \in \mathbb{N}$
  - output: does G contain a path w/ $\geq k$ non-repeating vertices?
  - "param" version of Hamiltonian Path

- simple $\binom{n}{k} k!$ alg by trying all k-tuples
- not FPT, so we use the following trick
  - choose a coloring $\chi: V \to [k]$ uniformly at random and find a path whose colors are pairwise disj

$\underbrace{\text{path whose colors are pairwise disj}}_{\text{"rainbow path"}}$

- suppose we can find this coloring in time $O^*(f(k))$, is this helpful?

LET STUDENTS PROVE ALL LEMMAS

Lem: $\exists$ k-path in LONGEST PATH $\iff$ $\exists$ COLORED LONGEST PATH

- $\Leftarrow$ trivial
- $\Rightarrow$
- let $u_1, \dots, u_k$ be a k-path
- color u_i w/ color i
- color the rest arbitrarily

Lem: $\exists$ alg for solving COLORED LONGEST PATH in time $O^*(2^k)$

- idea: find the path via DP where we keep track of colors used, last vertex of the path

- recursive fct $f(C \subseteq 2^{[k]}, u \in V)$:
  - C: set of colors used by current path
  - u: last vertex of the path being built
  - returns True iff there exists a path w/ one vertex of each color in C ending w/ u
  - if $|C| = 1$, and $c(u) \in C \to$ True
    $c(u) \notin C \to$ False
  - if $\chi(u) \notin C \to$ False
  - return $\bigvee_{c \in C} f(C \setminus c, v)$ where $uv \in E$

. now we know how to solve COLORED LONGEST PATH, all we need to do is to prove that many colorings are good. What is good?

Lem: Suppose we have a YES-instance w/ path $P = u_1, u_2, \ldots, u_k$. A random coloring will assign a different color to vertices of P w/ pr. $\geq e^{-k+1}$

Pf:

. # all colorings: k^n

. # colorings that give vertex of P a different color : $k!$

. we don't care about coloring of $V(G \setminus P)$: k^{n-k}

$$\frac{k!\, k^{n-k}}{k^n} = \frac{k!}{k} \geq \left(\frac{\frac{k^k}{e^{k-1}}}{k^k}\right) = \frac{1}{e^{k-1}}$$

Lem: There exists a randomized $O^*((2e)^k)$-time alg for LONGEST PATH w/ error rate $\leq \frac{1}{n^c}$.

Pf:

. color gr. randomly

. apply COLORED LONGEST PATH alg

. if it returns YES, return YES

. repeat the above for $c \lg n\, e^k$ iterations

. if none of the attempts succeed, return NO

. alg fails w/ pr $\leq 1 - \left(\frac{1}{e}\right)^{k-1} \leq e^{\left(\frac{1}{e}\right)^{k-1}}$

. repeat e^k times $\rightarrow \leq e^{\left(\frac{1}{e}\right)^{k}-1} \cdot e^k =$

. and $c \lg n$ repetitions of → to amplify to $\frac{1}{n^c}$ error rate

Markov bound: Let X be a positive r.v.. $Pr[X \geq a] \leq \frac{E[X]}{a}$ for any $a \in \mathbb{R}^+$

$1 + x \leq e^x \;\; \forall x \in \mathbb{R}$

$1 -$

$\left(\frac{1}{e}\right)^{\ln n} = e^{-\ln n}$

$n \, \emptyset$

$Pr[X \geq e^k]$

$\frac{1}{e} \quad \frac{1}{e}$

$\frac{e^{k-1}}{e^k} = \frac{1}{e}$

X : r. v.

e^k

$\left(\frac{1}{e}\right)^{(\ln n) \cdot 50} = n^{50}$

$n \, \emptyset \, \frac{1}{e}$