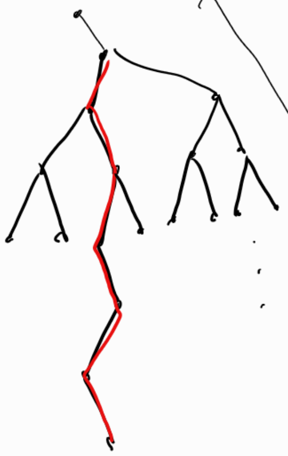
1.

every gr w/ deg≥3 has a cycle of size $\leq \lceil 2 \lg n \rceil$:

· proof by ↯ using BFS

· assume a shortest cycle $C = (u_1, \dots, u_k)$, $k > \lceil 2 \lg n \rceil$



· imagine we run BFS from $u_1$ in order to find C

· deg ≥ 3 -> $u_1$ has 2 more neighbors

· $u_i$ has $dist(u_i, u_1) = i-1$ because

· one can inductively prove, that there are $\geq 2^i$ vertices at dist $i$ from $u_1$ due to deg ≥3

· maybe I'll do it later

· but $\sum_{k=1}^{\lceil 2 \lg n \rceil + 1} 2^k > n$ ↯

-> there exists a cycle of len $\leq \lceil 2 \lg n \rceil$

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algo for FVS

· apply the rules below exhaustively prioritizing rules w/ small indices

rule 1: Remove all deg 1 vertices as the don't lie in cycles

rule 2: If $u$ has deg 2 w/ neighbors $v_1, v_2$, connect $v_1, v_2$ and remove

· the rationale here is that any FVS that contains $u$ can replace $u$ w/ either $v_1$ or $v_2$

· rule 2 might create multigraphs though, we need to remove them

rule -1: If we see a loop (edge $(u,u)$), remove it and decrease $k$ by 1.

·

rule 0: If we see parallel edges, keep two edges

- so rule 2 will always be applied on vertices $u$ where neighbors are different from $u$
- and if rule 2 creates a loop, rule 0 takes care of that

rule 5: $k < 0$, reject (sanity check)

→ we arrive at gr. w/ deg $\geq 3$

- we find the shortest cycle of len $\leq \lceil 2 \lg n \rceil$
- we branch on the vertex we delete from it

→ $O^*((\lg n)^k)$ alg

- is this an FPT alg?
- 2 cases: $n \leq k^k$ and $\lg n > k^k$
- first case gives: $O^*((\lg k^k)^k) = (k \lg k)^k = \lg k^{k \lg k} \in O^*(2^{O(k \lg k)})$
- second case: $n > k^k$

$$\cdot\ (\lg n)^k = 2^{k \lg\lg n} \leq 2^{\frac{k^2 + (\lg\lg n)^2}{2}} = 2^{\frac{k^2}{2}} \cdot 2^{\frac{(\lg\lg n)^2}{2}} = 2^{\frac{k^2}{2}} \cdot n^{o(1)}$$

Cauchy-Schwarz

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Clique (IS) $r$-regular graph

- if $k > r+1$, reject immediately as no vertex has enough neighbors
- so $k \leq r+1$
- easy XP alg checks every subset → $\binom{n}{k}\binom{n}{2} \leq n^{r+1} \cdot n^2 = n^{r+3}$ alg
- otherwise, for each vertex $u$ check if it and its neighbors form a large clique

$$n \cdot \binom{r}{k} k^2 = O(r^k k^2) \checkmark$$

Cluster Vertex Deletion

- recall that gr is a cluster gr $\iff$ it does not have a $P_3$ as an induced subgr from last tutorial class

- so for every cherry (= induced $P_3$) we try to branch on which vertex we kill

- to be exact, for each 

, we branch on
$G - u_1, k-1$
$G - u_2, k-1$
$G - u_3, k-1$

- 3 vertices $\rightarrow O^*(3^k)$ alg
  - specifically $T(n) = 3\,T(n-1)$
    $T(1) = 1$

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3. Counting vertex covers

- if deg $\leq 2$, G is paths and cycles $\rightarrow$ easy counting
- otherwise recurse on max degree vertex $u$

$$T(n) = T(n-1) + T(n-3)$$

add $u$ to VC          add its $> 2$ neighbors to VC

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$\mathcal{F}$-free gr. modification

- let $l$ be size of $\mathcal{F}$ as in sum of # vertices of grs in $\mathcal{F}$
- recall that $l \in O(1)$
- check in time $\binom{n}{l}$ if G is $\mathcal{F}$ free
- if yes, done
- o/w there are $2^l \in O(1)$ to break a copy of a gr. in $\mathcal{F}$

- branch
- runtime $O^*\left(2^{O(\ell \cdot k)}\right)$ worst case