1.

(i) => (ii)

- suppose λ is an eigenvalue of A
- let v be the corresponding eigenvector to λ
- by def. of an eigenvalue: $Av = \lambda v \quad / \cdot v^T$ from the left

$$v^T A v = \lambda v^T v$$

- by (i), $v^T A v \geq 0$
- $v^T v \geq 0$ always (even $v^T v > 0$, eigenvectors are nonzero)

$$\Rightarrow \lambda \geq 0$$

(ii) => (iii)

- let $\lambda_1, \dots, \lambda_n$ be eigenvalues of A and $v_1, \dots, v_n$ the corresponding eigenvectors
- linear algebra fact: eigenvalues of symmetric matrices are real
- linear algebra fact: eigenvectors $v_1, \dots, v_n$ are orthonormal
- let $Q = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$, $\Delta = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$

- $$\begin{aligned} Av_1 &= \lambda_1 v_1 \\ Av_2 &= \lambda_2 v_2 \\ &\vdots \end{aligned} \quad \Rightarrow \quad AQ = Q\Delta$$

- Q is ortogonal $\Rightarrow \quad Q^T Q = I = QQ^T \Rightarrow Q^T = Q^{-1}$
- $AQ = Q\Delta \quad / \cdot Q^{-1}$ from the right

$$A = Q \Delta Q^{-1}$$

$$D = \begin{pmatrix} \sqrt{\lambda_1} & & & 0 \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ 0 & & & \sqrt{\lambda_n} \end{pmatrix}$$

- let $D = \sqrt{\Delta}$ element-wise $\Rightarrow \quad DD^T = DD = \Delta$

- $A = Q \Delta Q^{-1} = Q D D^T Q^{-1} = Q D D^T Q^T = QD (QD)^T$

- the matrix U we are looking for is QD

(iii) => (i)

- let $x \in \mathbb{R}^n$
- $x^T A x = x^T \overset{(iii)}{\widetilde{U^T U}} x = \underbrace{(Ux)^T}_{= y^T} \underbrace{(Ux)}_{= y} \geq 0$ => $y^T y \geq 0$ always

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2.

- $\max \quad -x_{11}$

$x_{12} = 1$

$X \succeq 0$

- feasible solutions always look like $\begin{pmatrix} x_{11} & 1 \\ 1 & x_{22} \end{pmatrix}$
- when is this psd? Iff $x_{11}, x_{22} \geq 0$, $x_{11} x_{22} \geq 1$
- then $\sup \left\{ -x_{11} : x_{12} = 1, \begin{pmatrix} x_{11} & 1 \\ 1 & x_{22} \end{pmatrix} \succeq 0 \right\} = 0$
- but x_{11} cannot be zero

3.

- hint: write it as

$$\min \quad x_n$$

$$x_1 \geq 2$$

$$\begin{pmatrix} x_i & x_{i-1} \\ x_{i-1} & 1 \end{pmatrix} \succeq 0 \qquad \text{for } i = 2, \dots, n$$

- is this an SDP?

$$\begin{pmatrix} \begin{matrix} x_n & x_{n-1} \\ x_{n-1} & 1 \end{matrix} & & & 0 \\ & \ddots & & \\ & & \begin{matrix} x_2 & x_1 \\ x_1 & 1 \end{matrix} & \\ 0 & & & \begin{matrix} x_1 & 2 \\ 2 & 1 \end{matrix} \end{pmatrix} \succeq 0$$

- yes

4.

- the most interesting case is when P is symmetric
- if it is not psd, then it has a negative eigenval λ w/ its eigenvector v
- then $(v^T v) \circ Y = v^T Y v \geq 0$ is the hyperplane

element-wise product

- if P is not symmetric, then $P_{ij} > P_{ji}$ for some i, j

- then $y_{ij} \leq y_{ji}$ is the hyperplane
- or finally, some linear constraint may be violated
- then that eqn gives the hyperplane