

(Near-)Optimal Algorithms for Sparse Separable Convex Integer Programs

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IP and ILP

Integer (Linear) Programming problem in standard form:

$$\min \{ \mathbf{w}^T \mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}, \mathbf{x} \in \mathbb{Z}^n \}, \text{ and} \quad (\text{ILP})$$

$$\min \{ f(\mathbf{x}) \mid A\mathbf{x} = \mathbf{b}, \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}, \mathbf{x} \in \mathbb{Z}^n \} \quad (\text{IP})$$

with

- ▶ $A \in \mathbb{Z}^{m \times n}$,
- ▶ $\mathbf{l}, \mathbf{u}, \mathbf{w} \in \mathbb{Z}^n$, and
- ▶ $f: \mathbb{R}^n \rightarrow \mathbb{R}$ a **separable convex function**, i.e. f is expressible as $f(\mathbf{x}) = \sum_{i=1}^n f_i(x_i)$ with each $f_i: \mathbb{R} \rightarrow \mathbb{R}$ convex.

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ILP is already **NP-hard** $\Rightarrow$ focus on **tractable subclasses** corresponding to **block-structured matrices**.

Block Structured Matrices

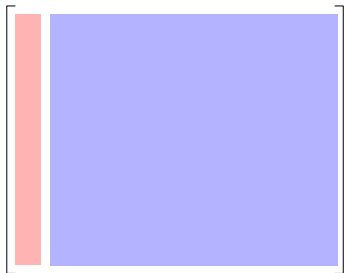
$$\begin{bmatrix} & & \\ & & \\ & & \\ & & \\ & & \end{bmatrix}$$

Multistage-stochastic matrix.

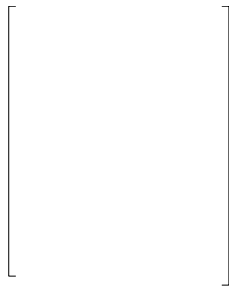
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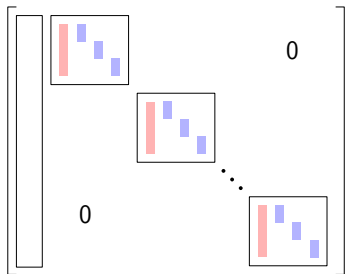
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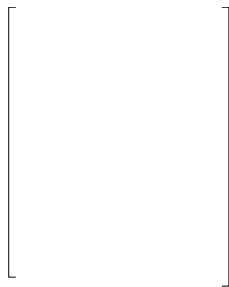
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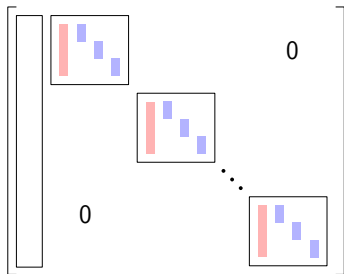


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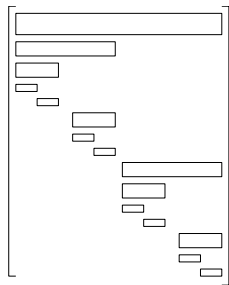


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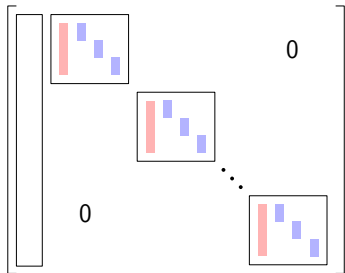


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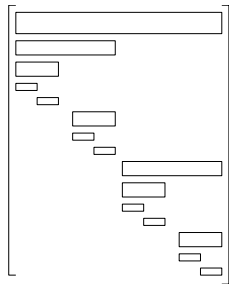


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- **Treedepth** measures the similarity of a graph to a star.

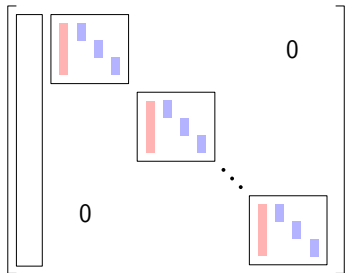


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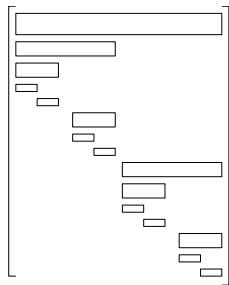


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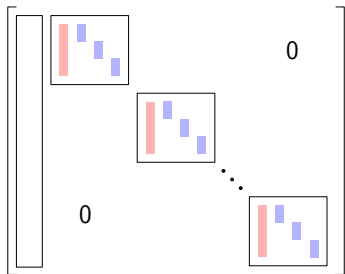


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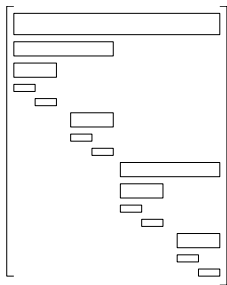


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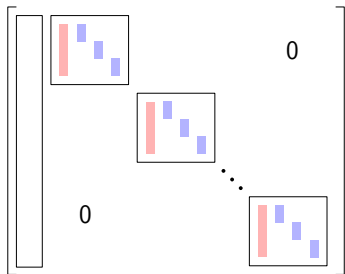


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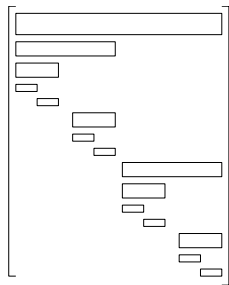
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Main point. Block structure-ness of $A \approx$ **treedepth**.



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$$g(\|A\|_\infty, d) (n^\omega + \min(m, n)nm) \log \|\mathbf{u} - \mathbf{l}\|_\infty \log f_{\text{gap}},$$

where $d = \min\{\text{td}_P(A), \text{td}_D(A)\}$, $f_{\text{gap}} = \max_{\mathbf{x}, \mathbf{y}: \mathbf{l} \leq \mathbf{x}, \mathbf{y} \leq \mathbf{u}} f(\mathbf{x}) - f(\mathbf{y})$.

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$\min f = \sum f_i \Leftrightarrow \min f_i$ separately $\Leftarrow \Omega(\log(u_i - l_i))$ comparisons
by **information theory**.

Our Results: Matching the Lower Bound

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There is an algorithm which solves IP in time

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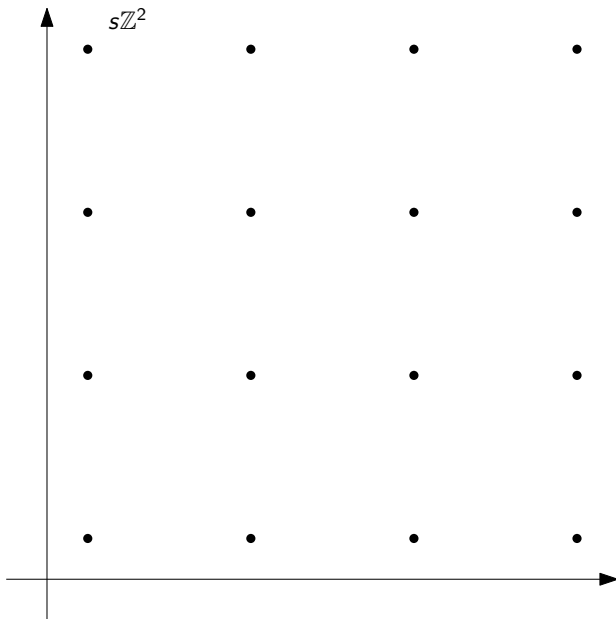
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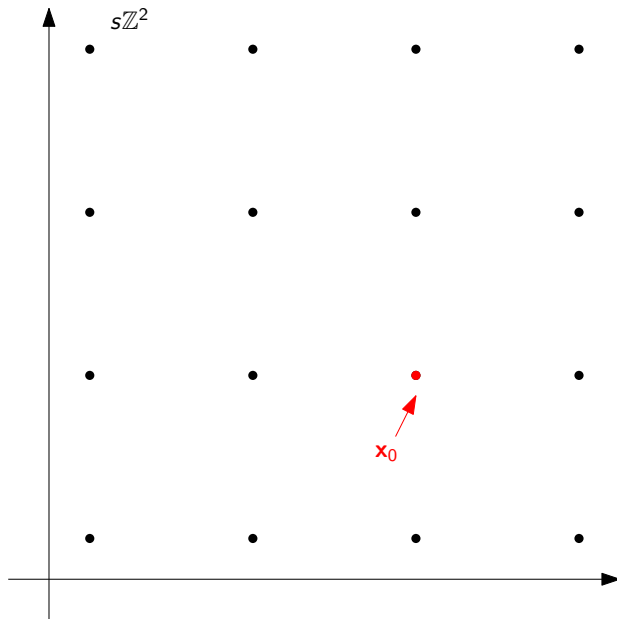
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- ▶ $\log \|\mathbf{b}\|_\infty \Leftarrow$ finding an **initial feasible solution**.
- ▶ **Conjecture:** The dependence on n in Theorem 2 is **optimal**.
- ▶ **Remark:** optimizing dependence on parameters $\|A\|_\infty, \text{td}_P(A), \text{td}_D(A)$: **NOT** focus of this work.

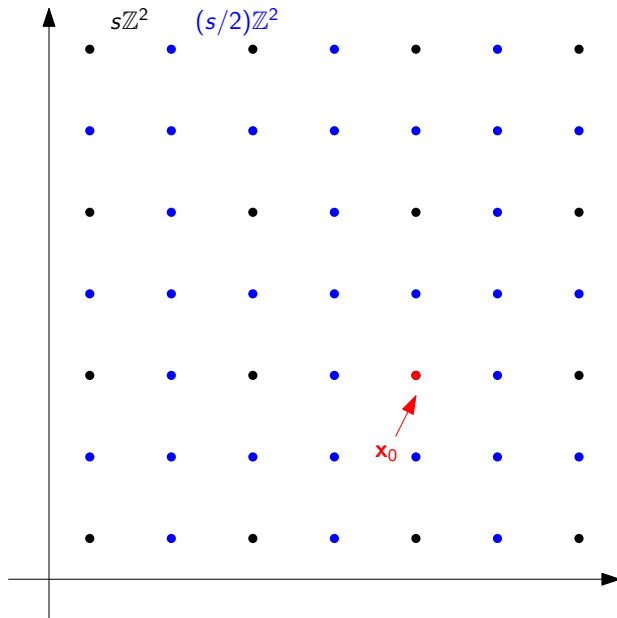
Main Idea: Scaling Algorithm



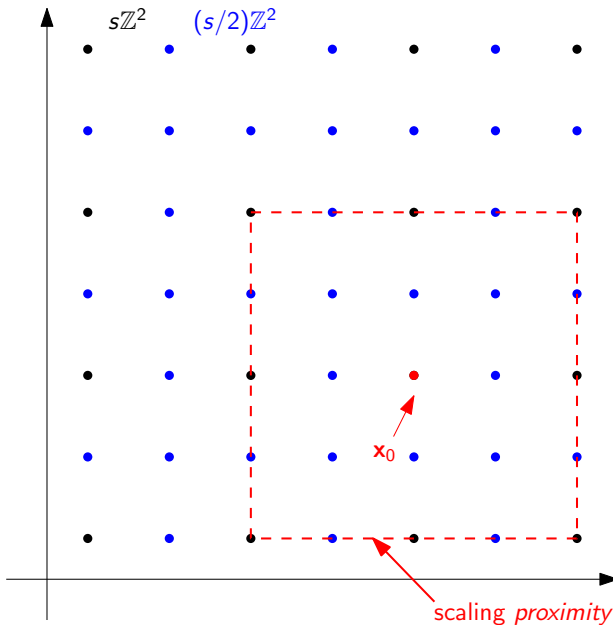
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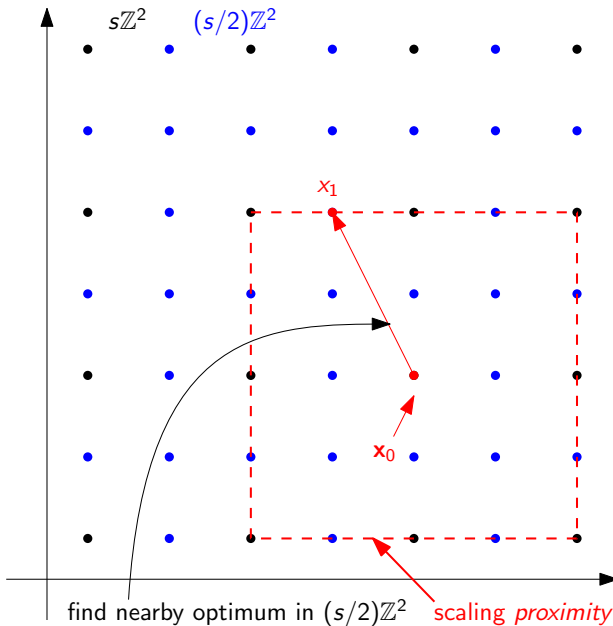
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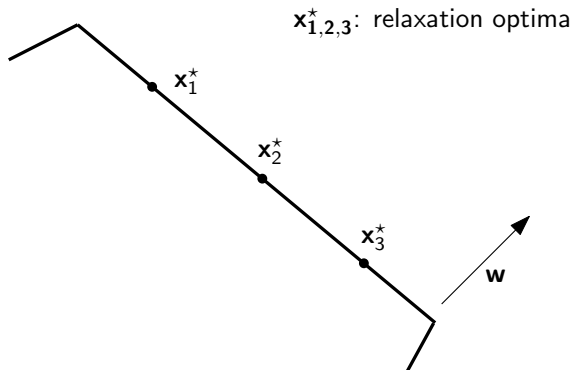
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Question. What is ρ ?

Conformal Proximity Bound (CPB) (very informally)

$\mathcal{P}_p(A)$: given A , smallest real $r \in \mathbb{R}$ such that

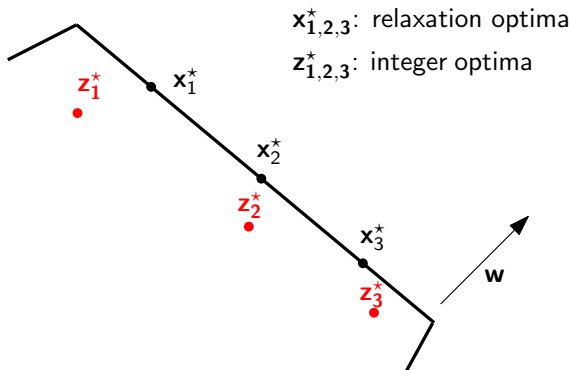
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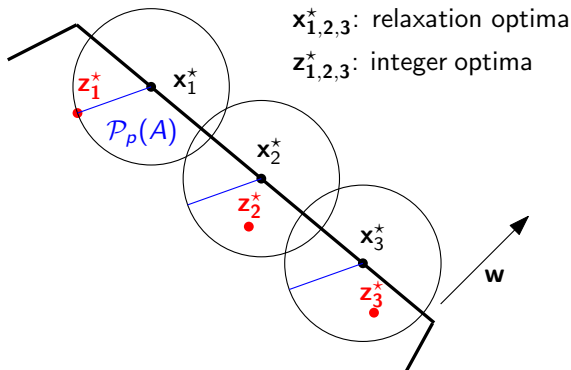
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- ▶ with $\|\mathbf{x}^* - \mathbf{z}^*\|_p \leq r$.



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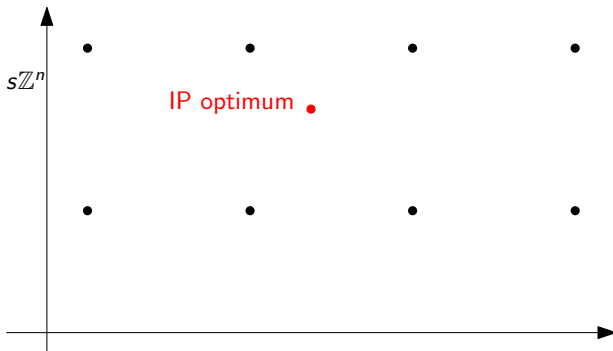
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- ▶ then there exists a solution $\mathbf{z}'$ of $\mathcal{I}'$ with

$$\|\mathbf{z}^* - \hat{\mathbf{z}}\|_p \leq (s+1)\mathcal{P}_p(A),$$

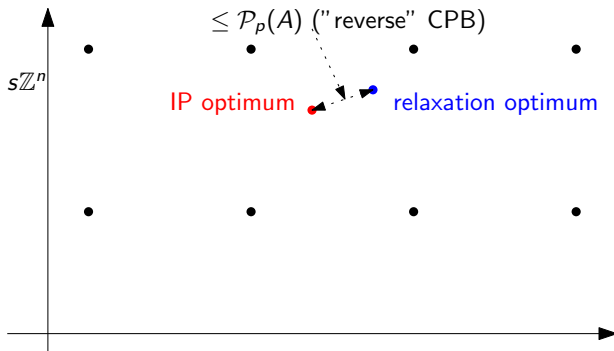
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Goal: bound distance between IP optimum and optimum restricted to $s\mathbb{Z}^n$



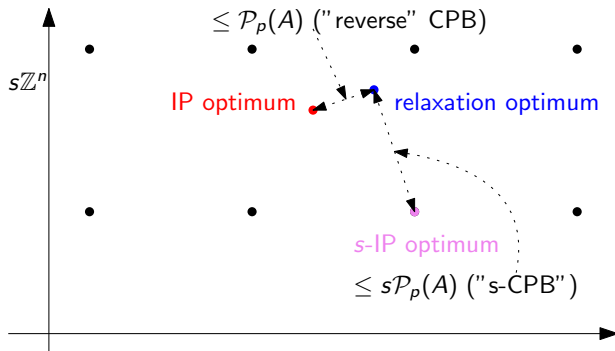
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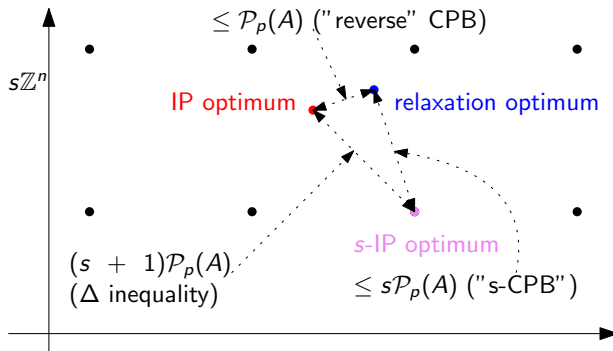
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Remark: Number of IP instances can be **reduced** to $\log \lceil \|\mathbf{l}, \mathbf{u}\|_\infty / \rho \rceil + 1$.

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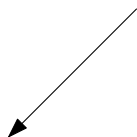
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Dual Algorithm (Theorem 2)

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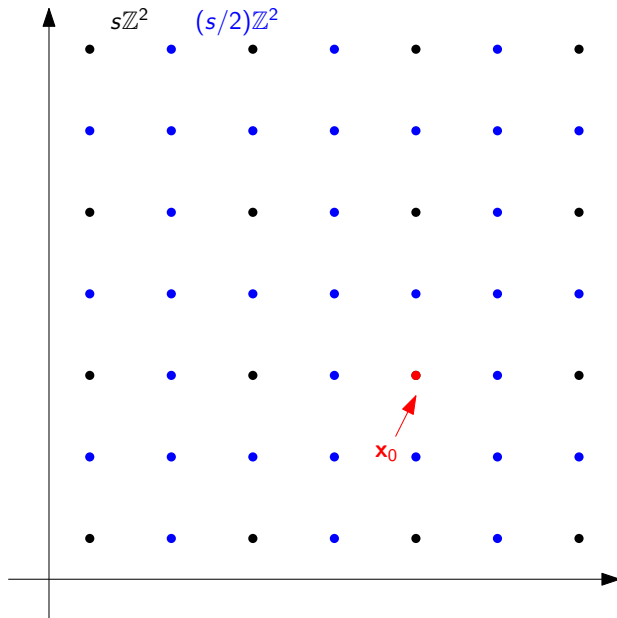
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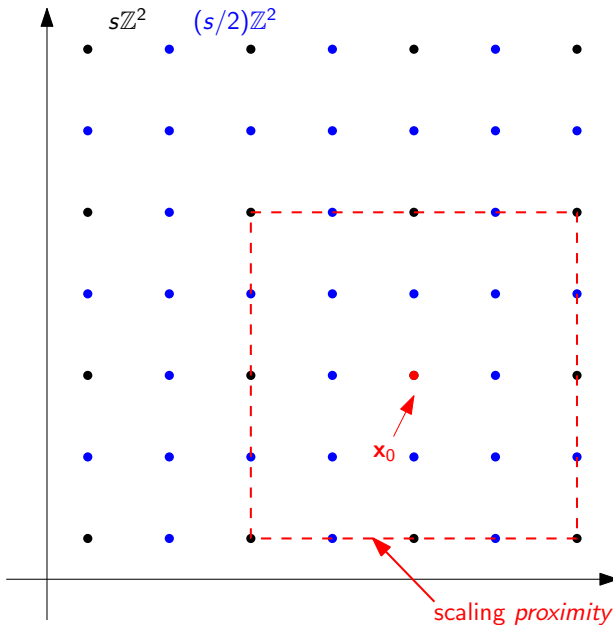
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- ▶ [Cslovjcek, Eisenbrand, Hunkenschröder, Rohwedder, Weismantel; SODA 2021, Proposition 4.1] shows instances of IP with
 - ▶ small $\text{td}_D(A) + \|A\|_\infty$
 - ▶ but where integer optima are $\Omega(n)$ far in the ℓ_∞ -norm from any continuous optimum.

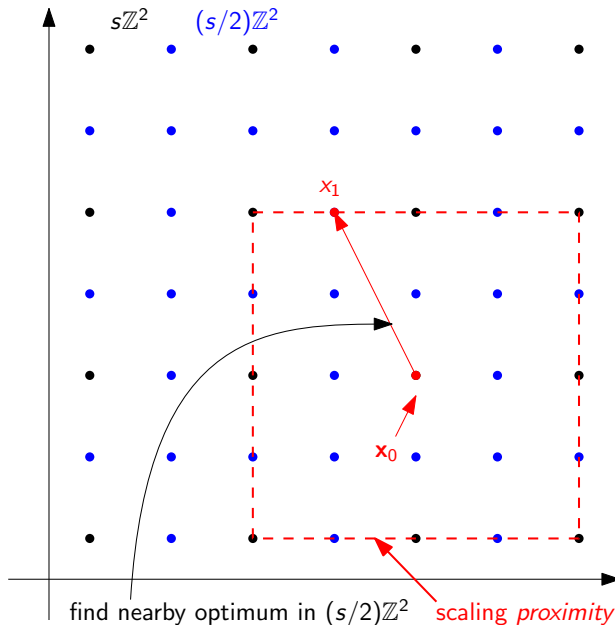
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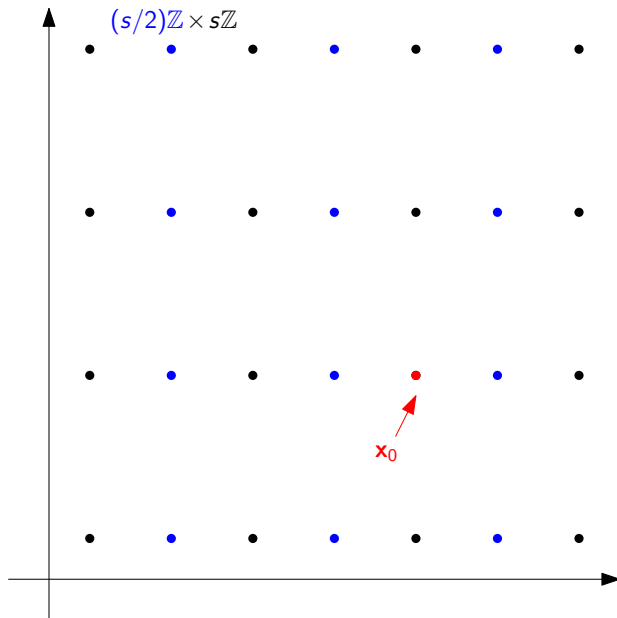
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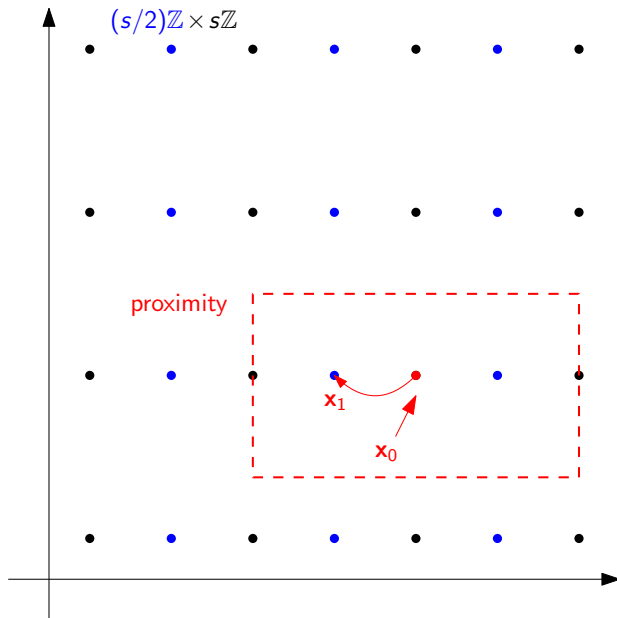
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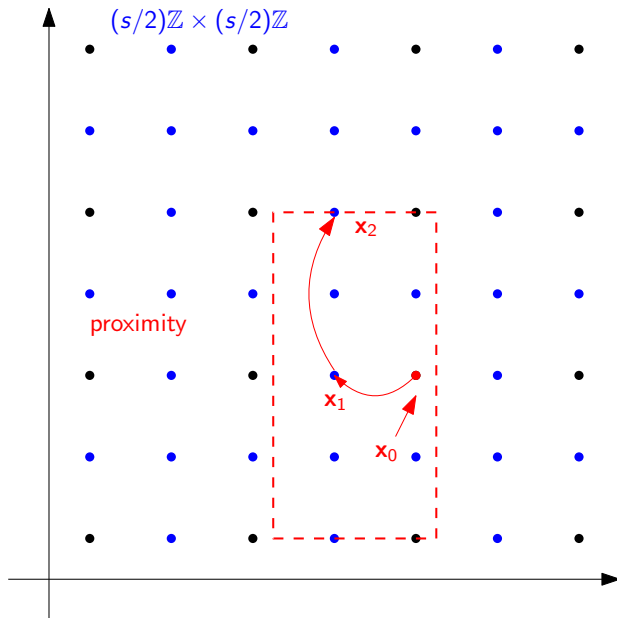
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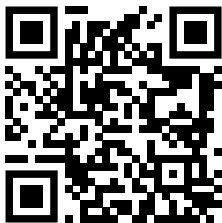
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Thank you for your attention!



<https://arxiv.org/abs/2505.22212>



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