(Near-)Optimal Algorithms for Sparse Separable Convex Integer Programs

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IP and ILP

Integer (Linear) Programming problem in standard form:

$$\min\left\{\mathbf{w}^{\mathsf{T}}\mathbf{x} \mid A\mathbf{x} = \mathbf{b},\, \mathbf{l} \leq \mathbf{x} \leq \mathbf{u},\, \mathbf{x} \in \mathbb{Z}^n\right\}, \text{ and} \qquad \text{(ILP)}$$

$$\min\left\{f(\mathbf{x}) \mid A\mathbf{x} = \mathbf{b},\, \mathbf{l} \leq \mathbf{x} \leq \mathbf{u},\, \mathbf{x} \in \mathbb{Z}^n\right\} \qquad \text{(IP)}$$

with

- $A \in \mathbb{Z}^{m \times n}$,
- $\mathbf{l}, \mathbf{u}, \mathbf{w} \in \mathbb{Z}^n$, and
- $f : \mathbb{R}^n \to \mathbb{R}$ a separable convex function, i.e. f is expressible as $f(\mathbf{x}) = \sum_{i=1}^{n} f_i(x_i)$ with each $f_i : \mathbb{R} \to \mathbb{R}$ convex.

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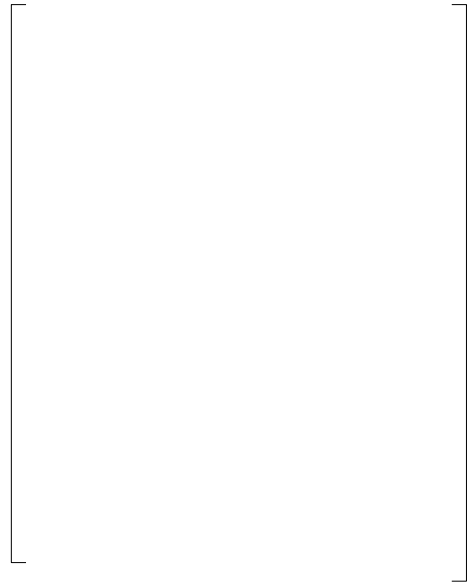
ILP is already NP-hard $\Rightarrow$ focus on tractable subclasses corresponding to block-structured matrices.

Block Structured Matrices

Multistage-stochastic matrix.



Tree-fold matrix.

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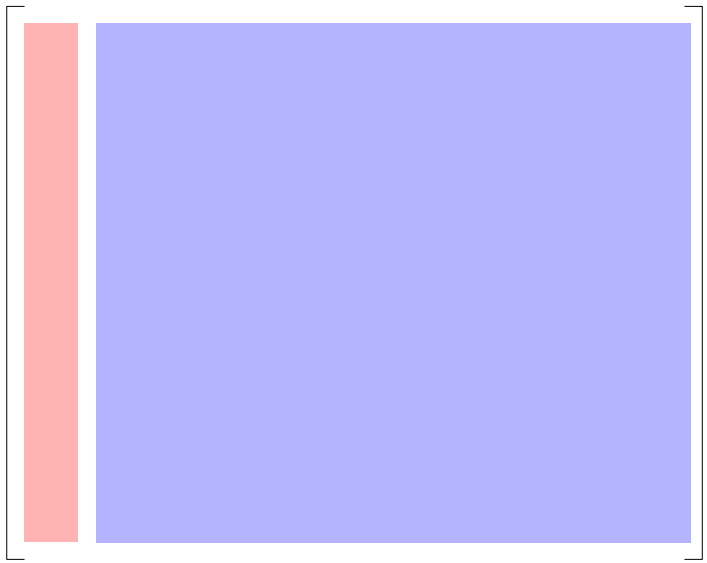

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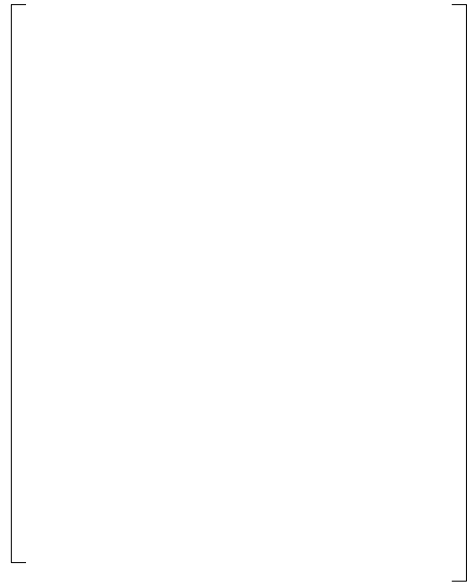

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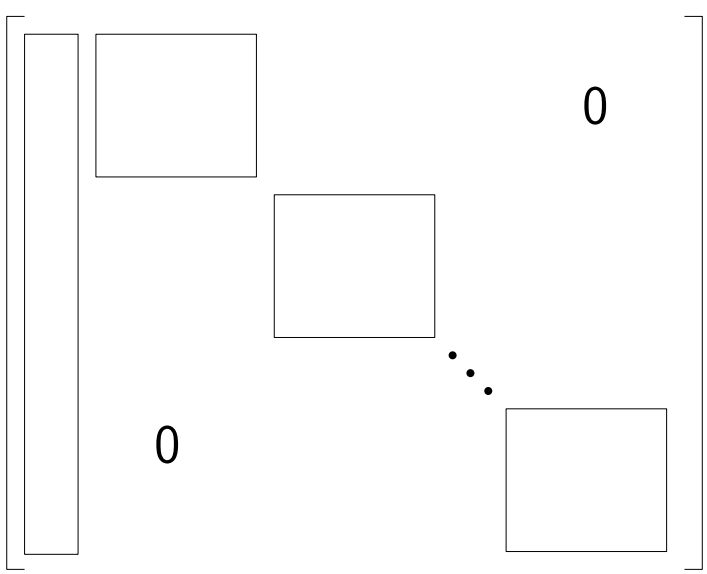

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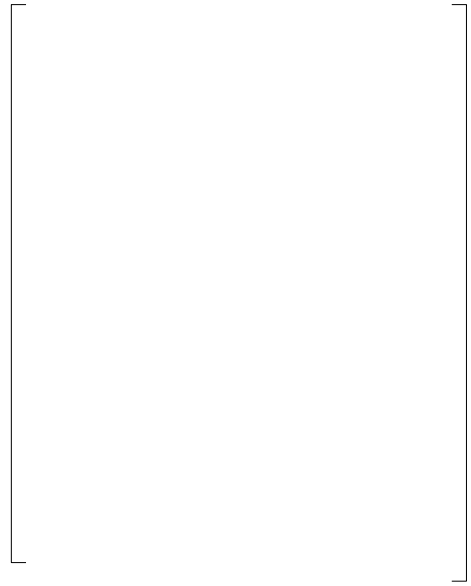

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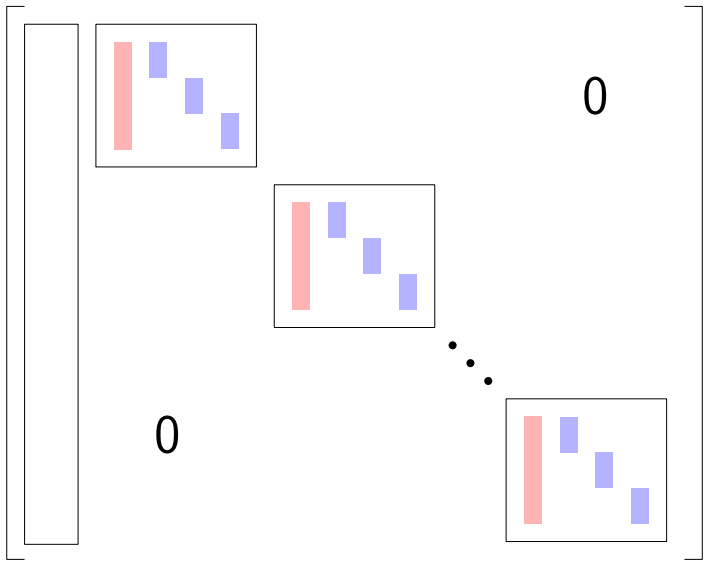

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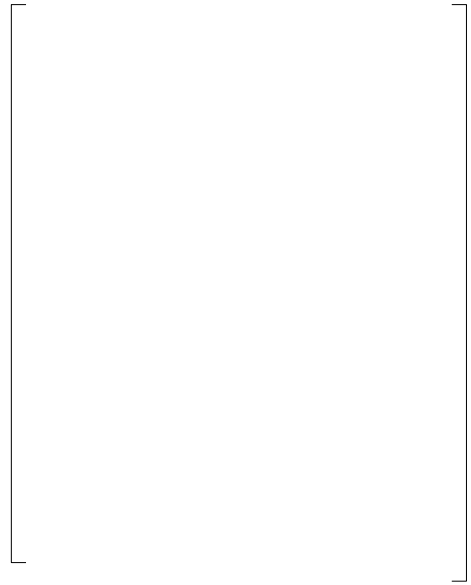

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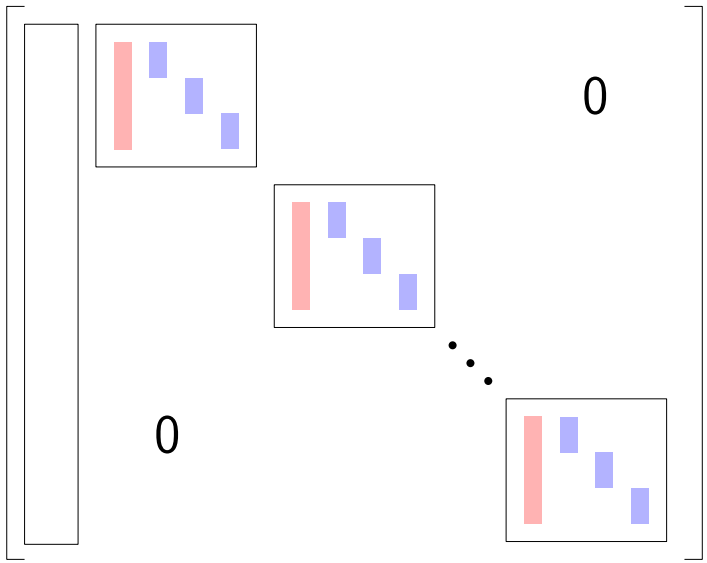

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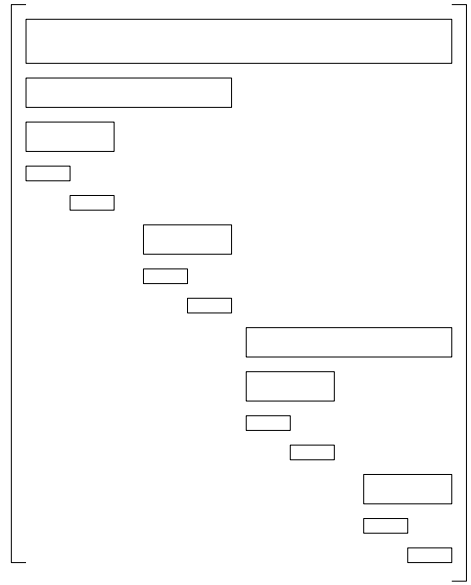

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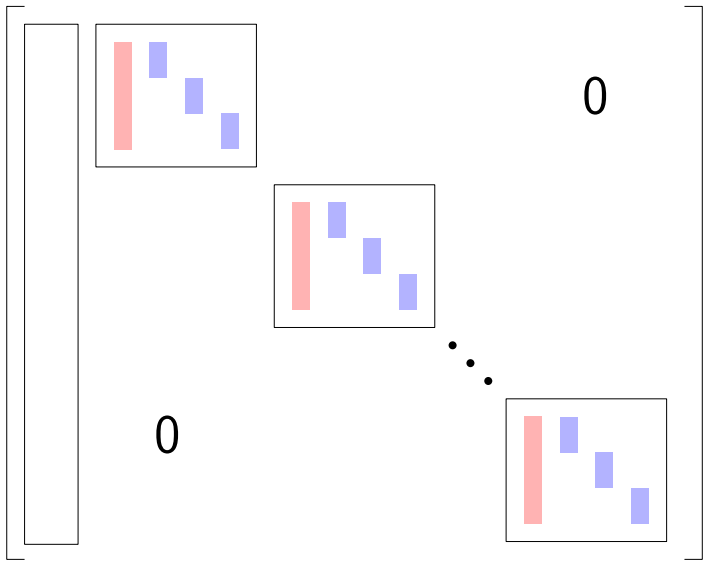

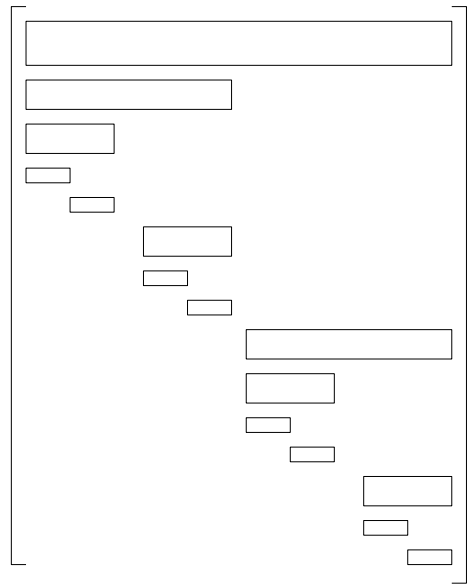

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- Treedepth measures the similarity of a graph to a star.

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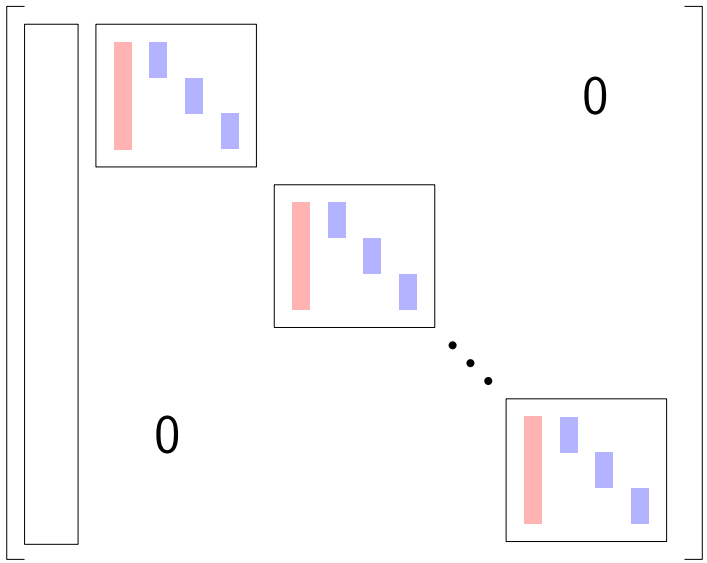

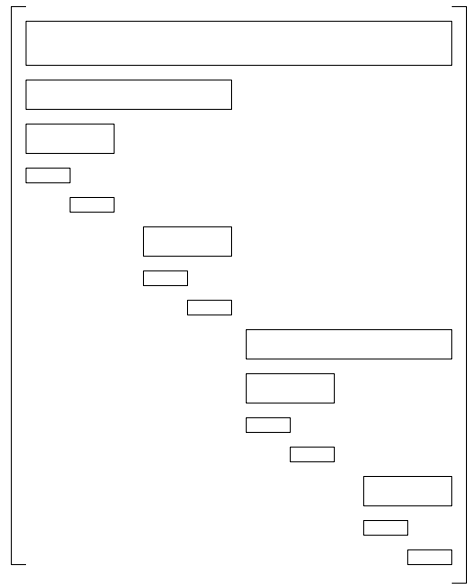

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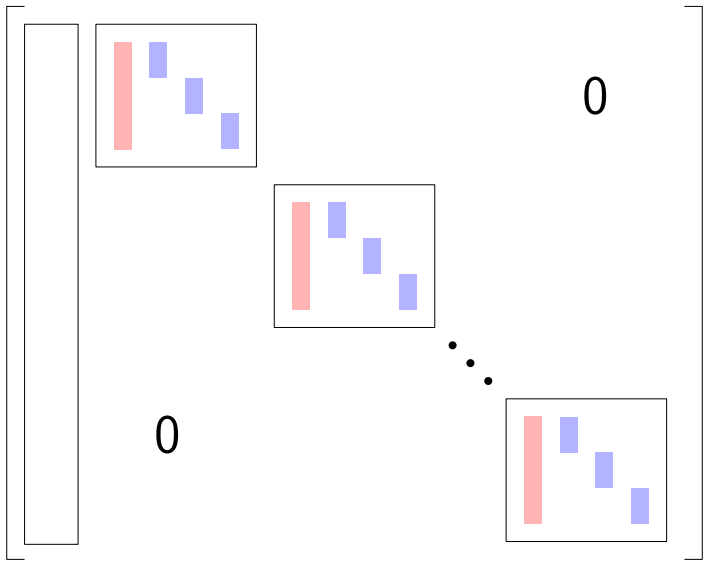

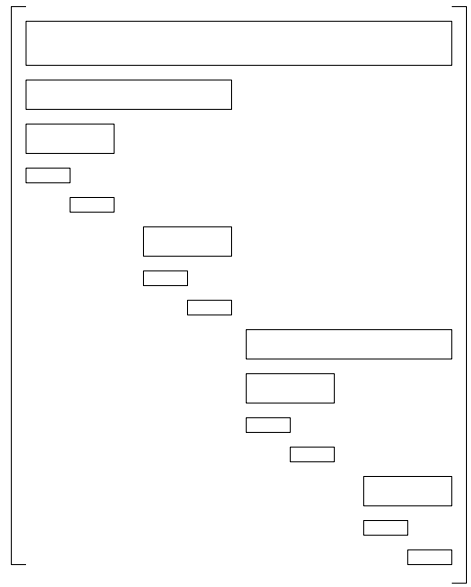

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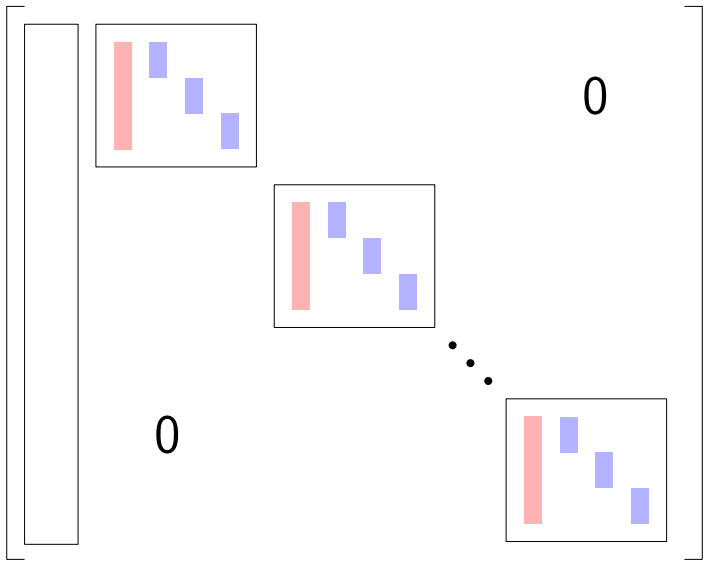

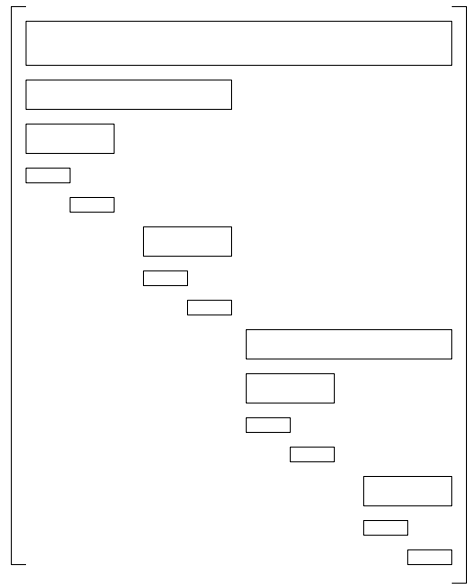

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Main point. Block structure-ness of $A \approx$ treedepth.

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$g, g', g'', \ldots$: some computable function.

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[Eisenbrand, Hunkenschröder, Klein, Kouteckỳ, Levin, Onn; MOR 2024]

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$$g(\|A\|_\infty, d) \; (n^\omega + \min(m,n)nm) \; \log \|\mathbf{u} - \mathbf{l}\|_\infty \log f_{\text{gap}} \;,$$

where $d = \min\{\text{td}_P(A), \text{td}_D(A)\}$, $f_{\text{gap}} = \max_{\mathbf{x},\mathbf{y}:\, \mathbf{l} \le \mathbf{x},\mathbf{y} \le \mathbf{u}} f(\mathbf{x}) - f(\mathbf{y})$.

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$\min f = \sum f_i \Leftrightarrow \min f_i$ separately $\Leftarrow \Omega(\log(u_i - l_i))$ comparisons by information theory.

Our Results: Matching the Lower Bound

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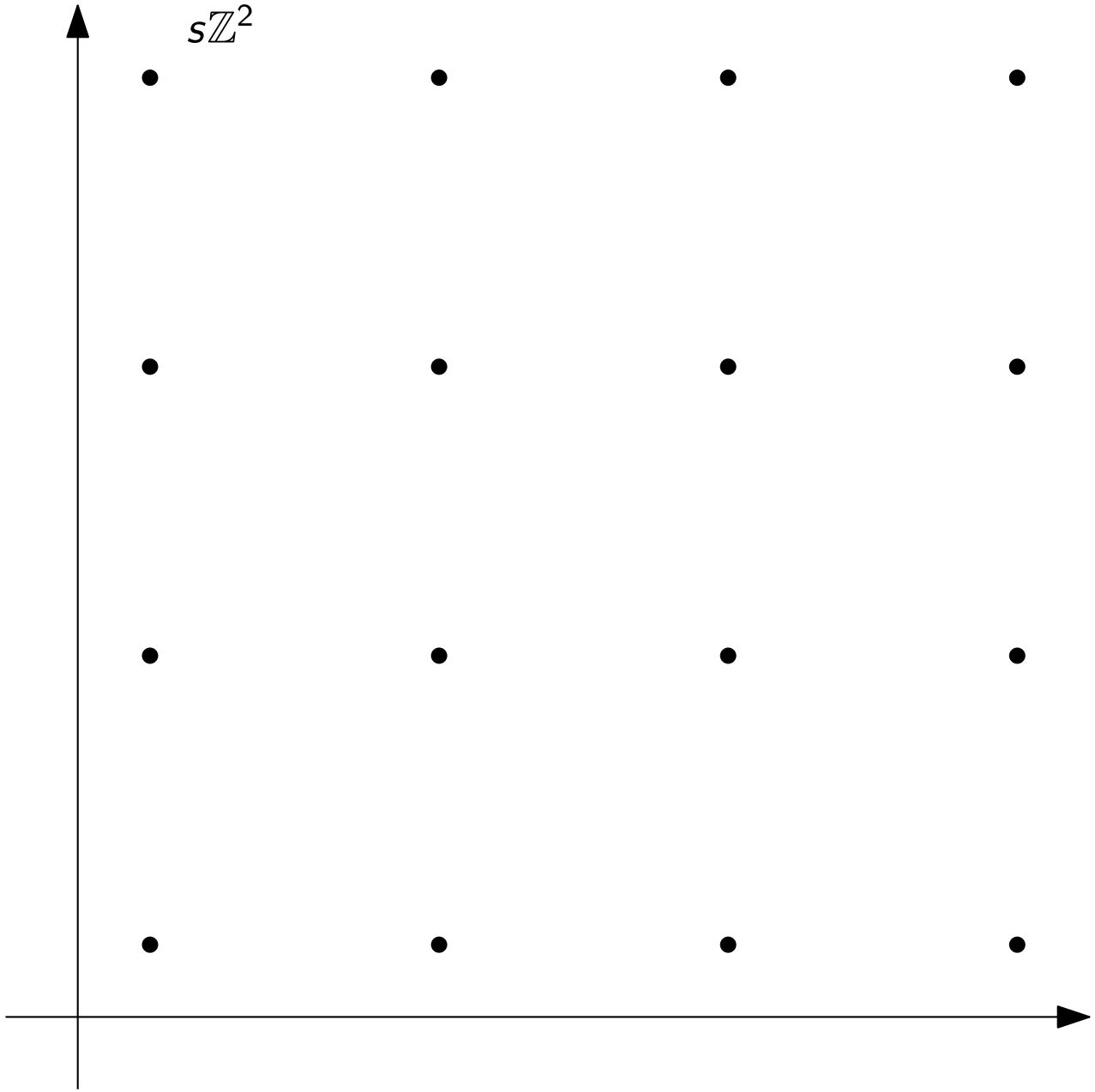
- **Remark:** optimizing dependence on parameters $\|A\|_\infty, \mathrm{td}_P(A), \mathrm{td}_D(A)$: **NOT** focus of this work.

Main Idea: Scaling Algorithm



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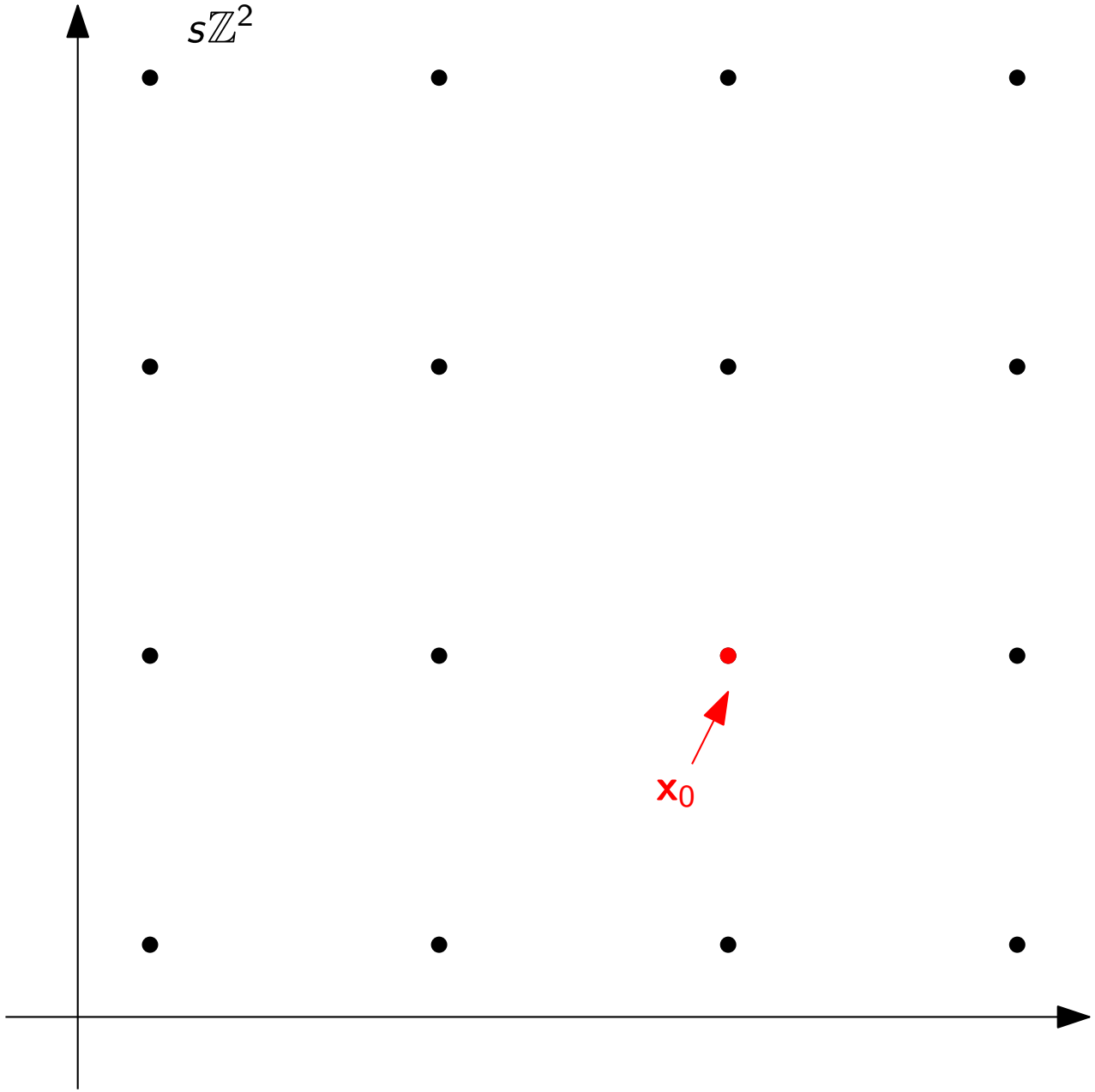

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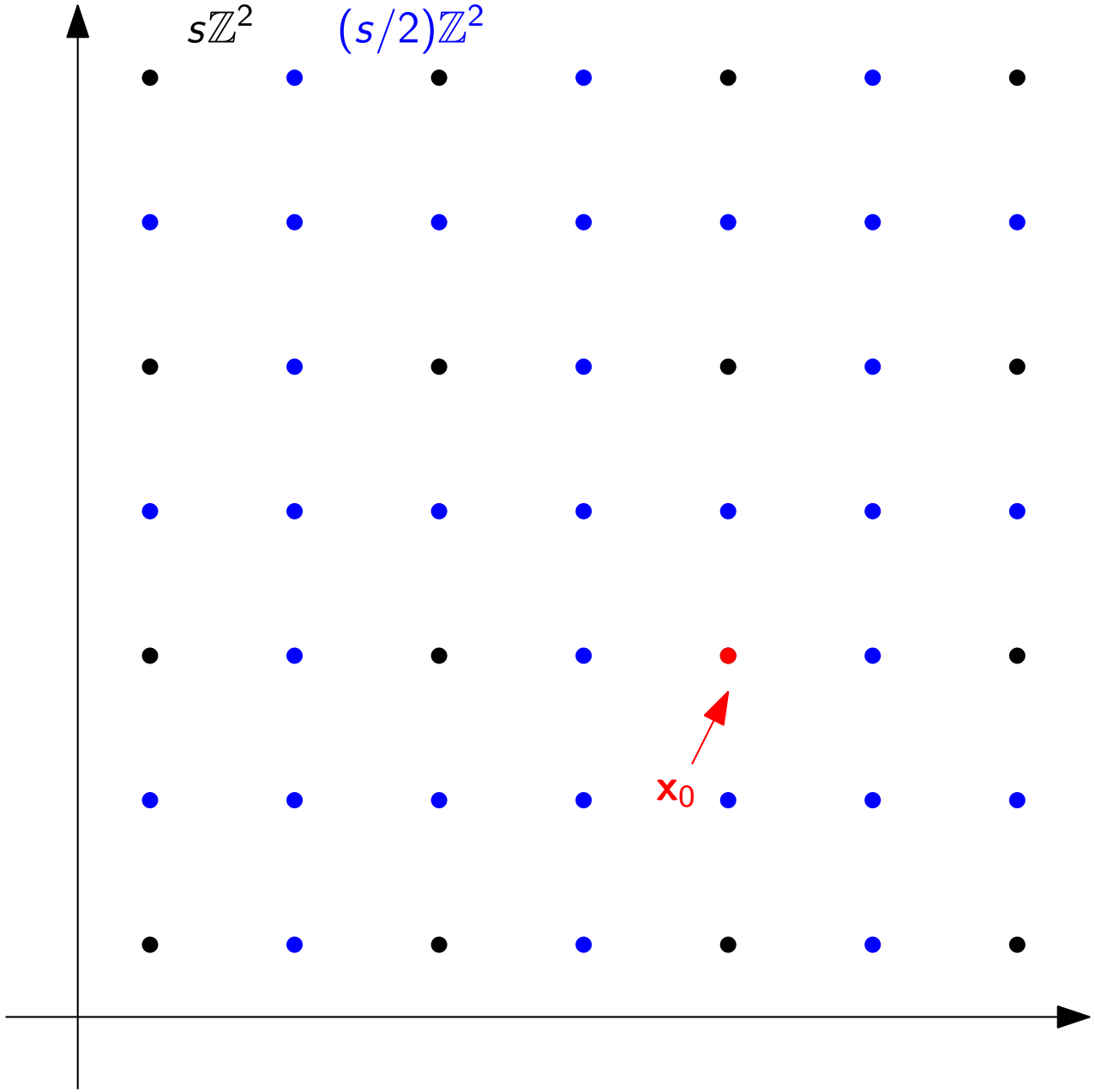

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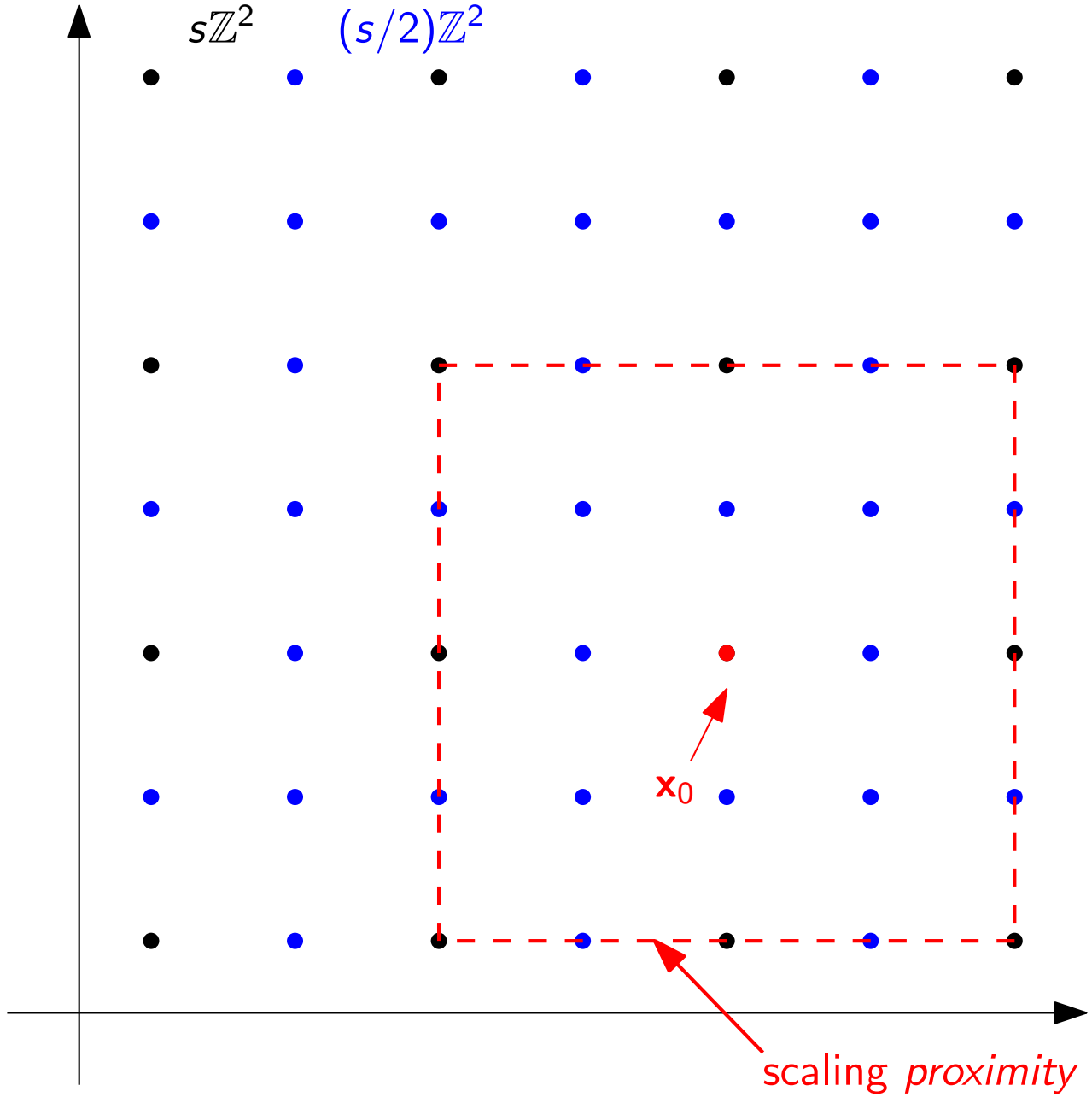

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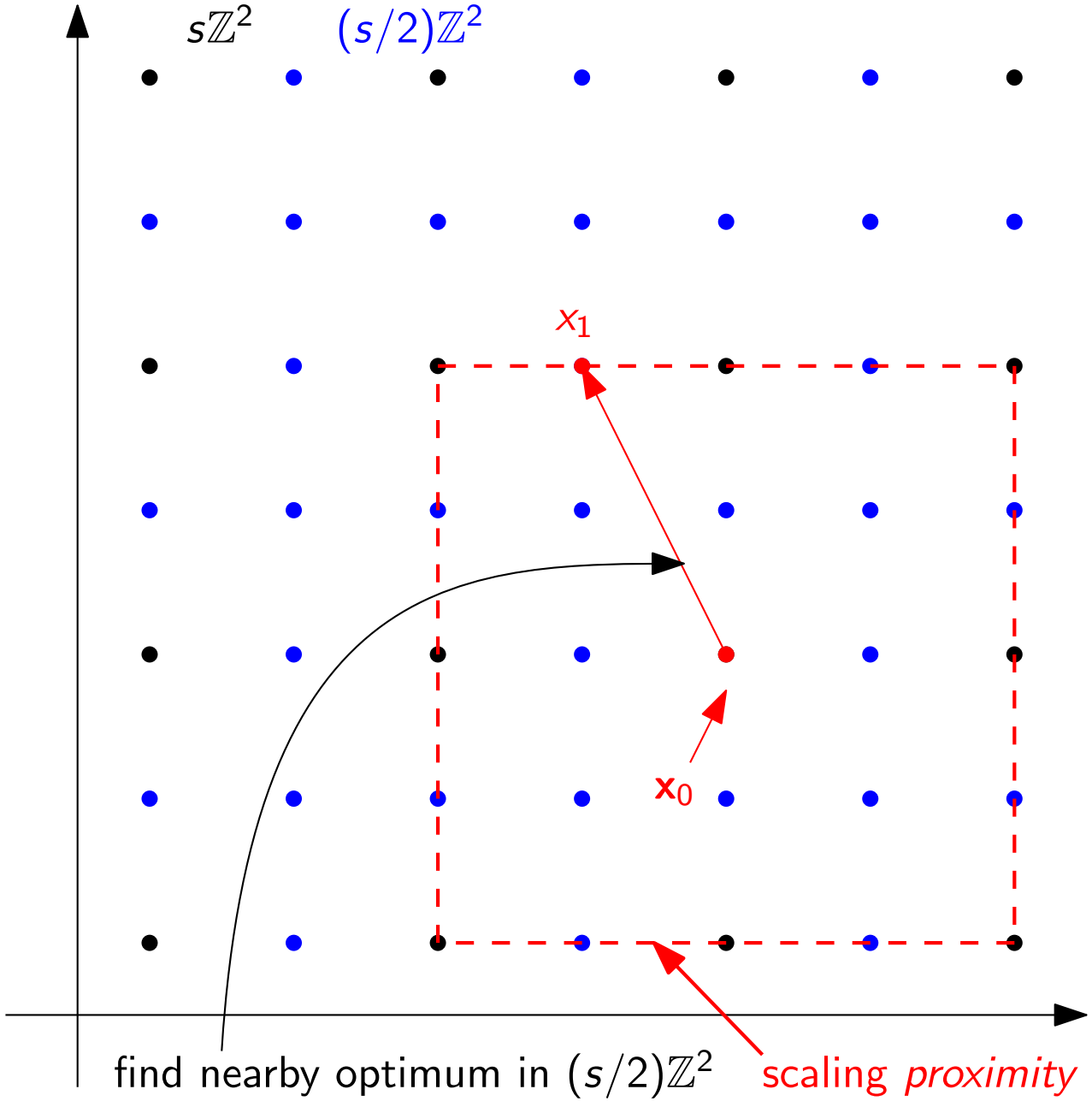

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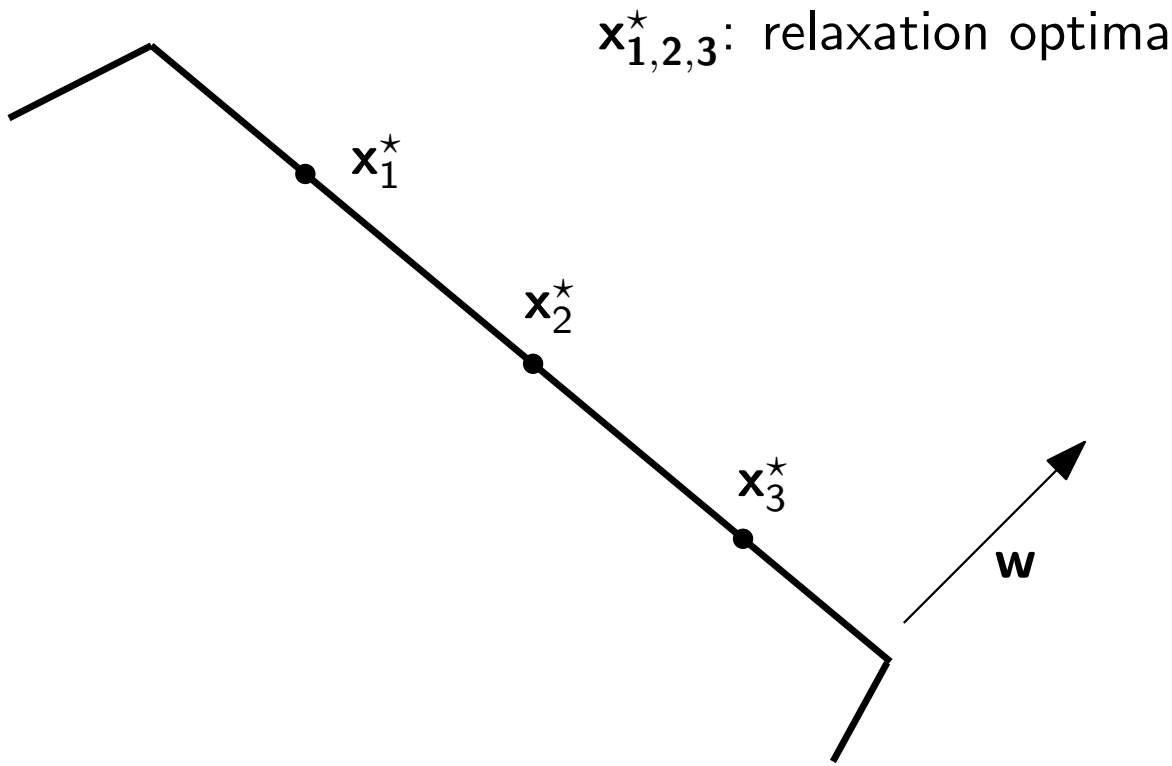
Question. What is ρ?

Conformal Proximity Bound (CPB) (very informally)

$\mathcal{P}_p(A)$: given A, smallest real $r \in \mathbb{R}$ such that

- for every relaxation optimum $\mathbf{x}^\star$



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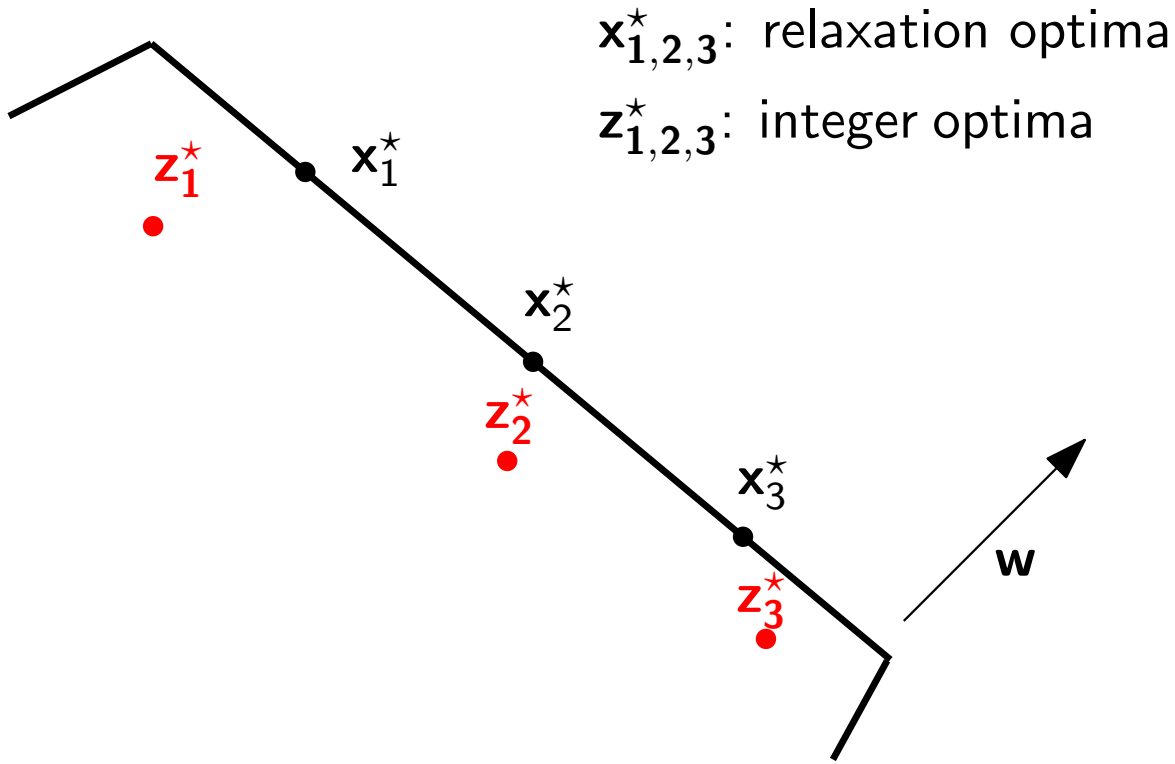

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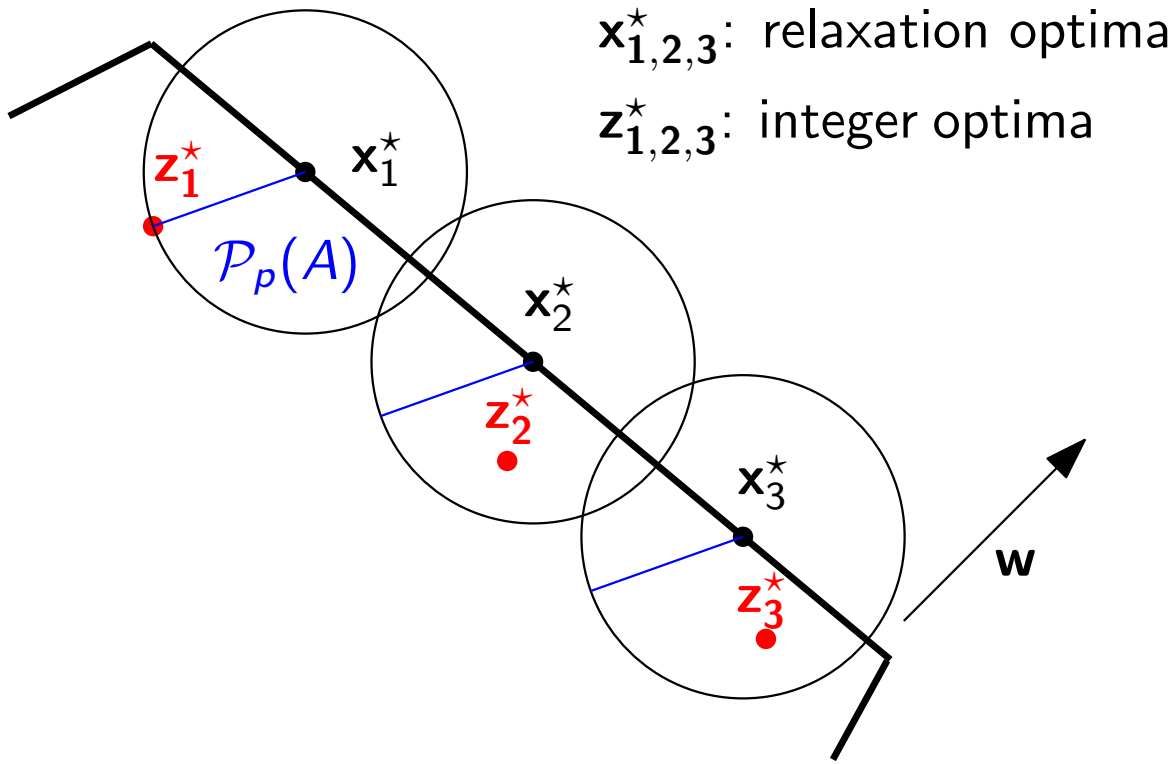
- with $\|\mathbf{x}^\star - \mathbf{z}^\star\|_p \leq r$.



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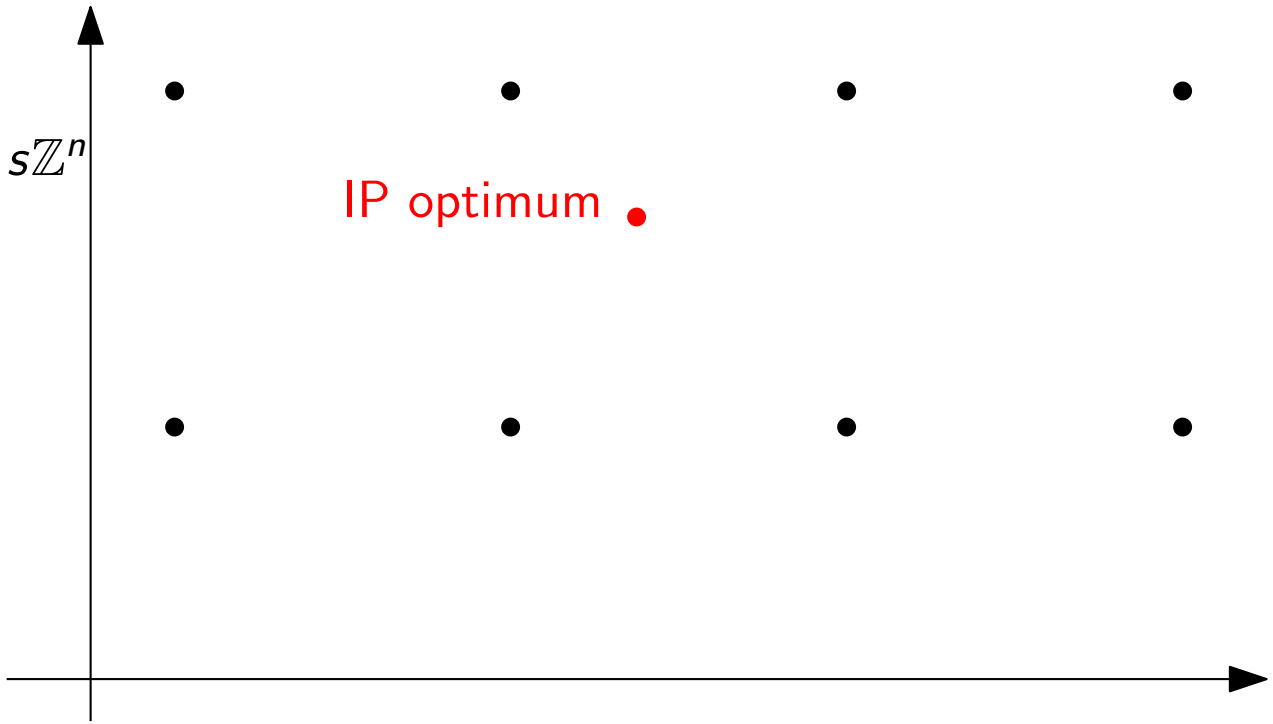
- $\mathcal{I}'$: *copy of* $\mathcal{I}$ *but require that solutions be in* $s\mathbb{Z}^n$
- *then there exists a solution* $\mathbf{z}'$ *of* $\mathcal{I}'$ *with*

$$\|\mathbf{z}^\star - \hat{\mathbf{z}}\|_p \leq (s+1)\mathcal{P}_p(A),$$

Scaling Proximity Theorem: Proof Idea

Goal: bound distance between IP optimum and optimum restricted to $s\mathbb{Z}^n$



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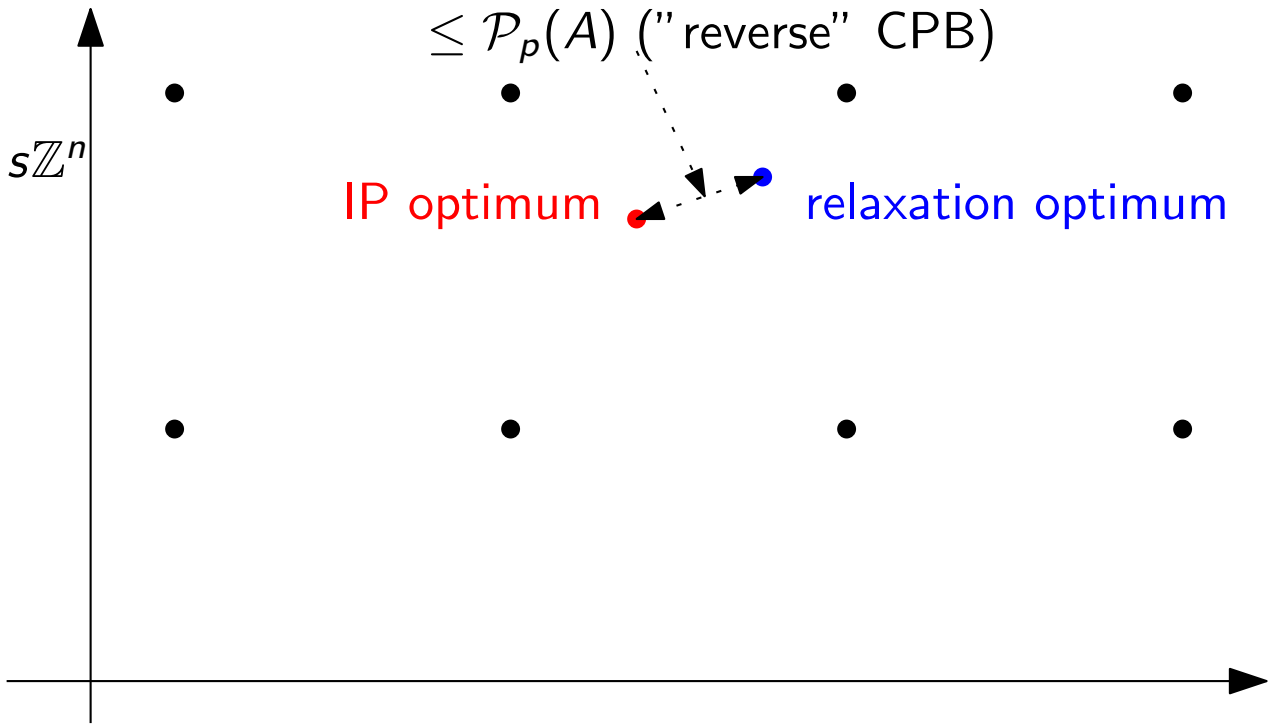

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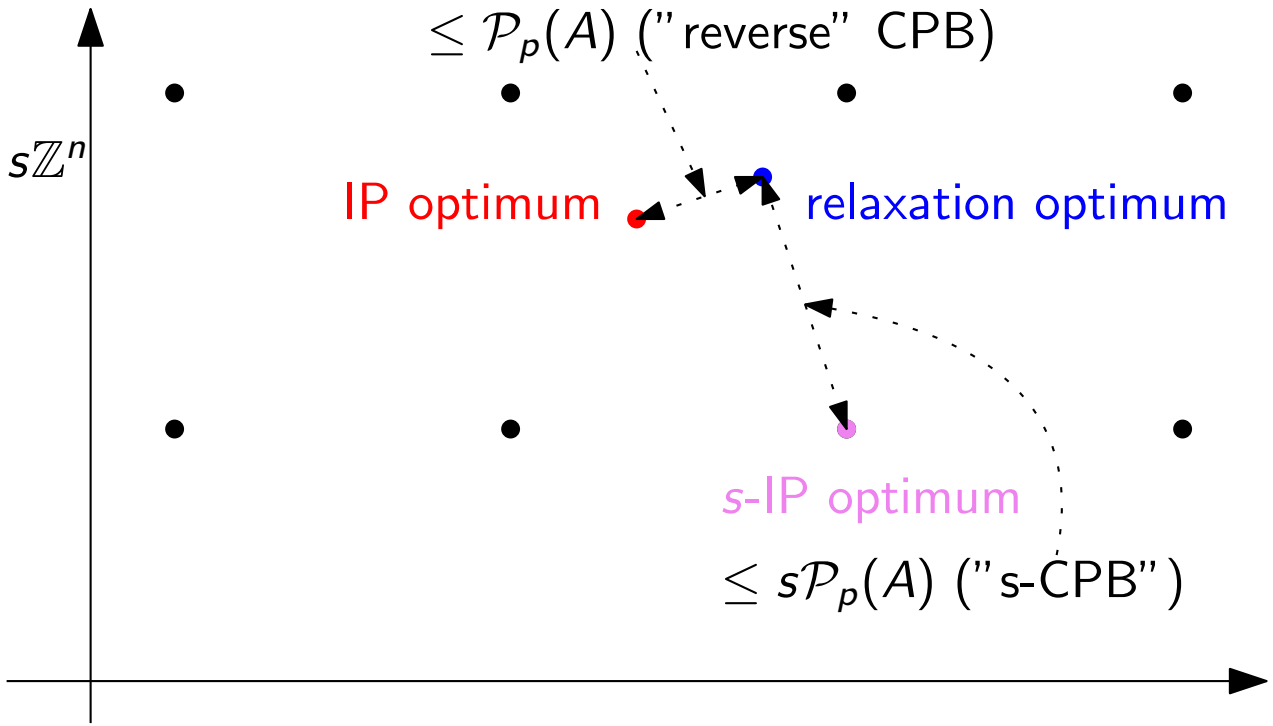

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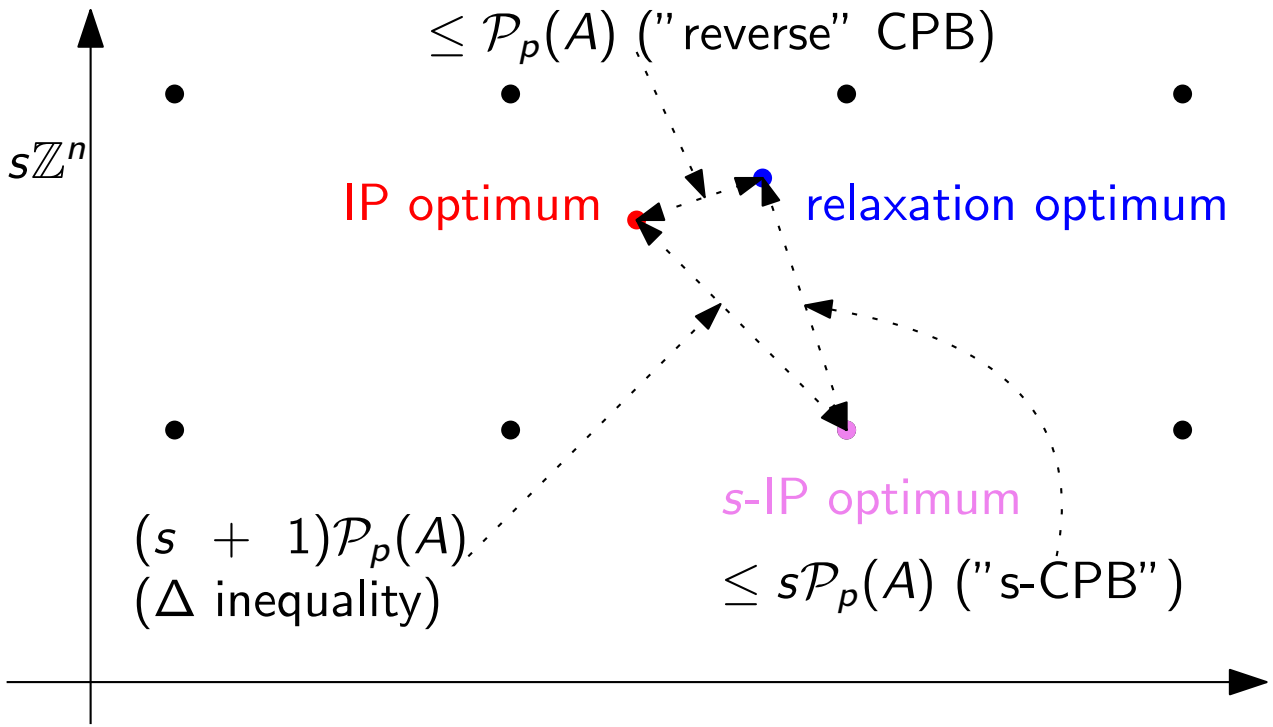

Scaling Algorithm: Practical takeaway

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Remark: Number of IP instances can be reduced to $\log \lceil \|\mathbf{l}, \mathbf{u}\|_\infty / \rho \rceil + 1$.

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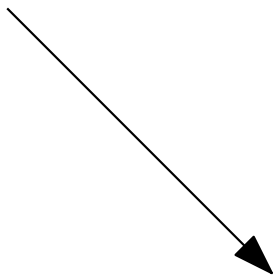

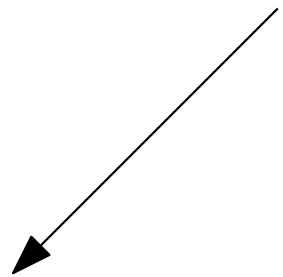

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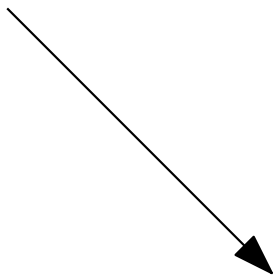

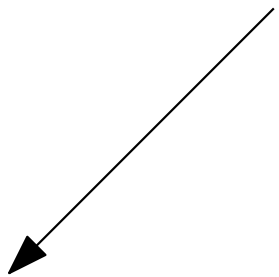

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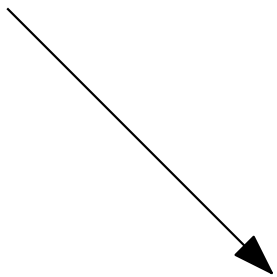

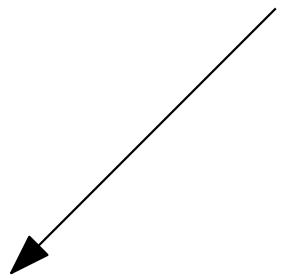

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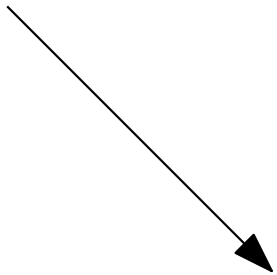

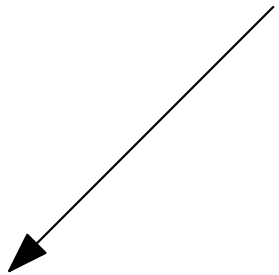

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- g' is triple exponential in $\mathrm{td}_P(A)$.
  - Seems optimal [Hunkenschröder, Klein, Koutecký, Lassota, Levin; IPCO 2024, Theorem 1]

Dual Algorithm (Theorem 2)

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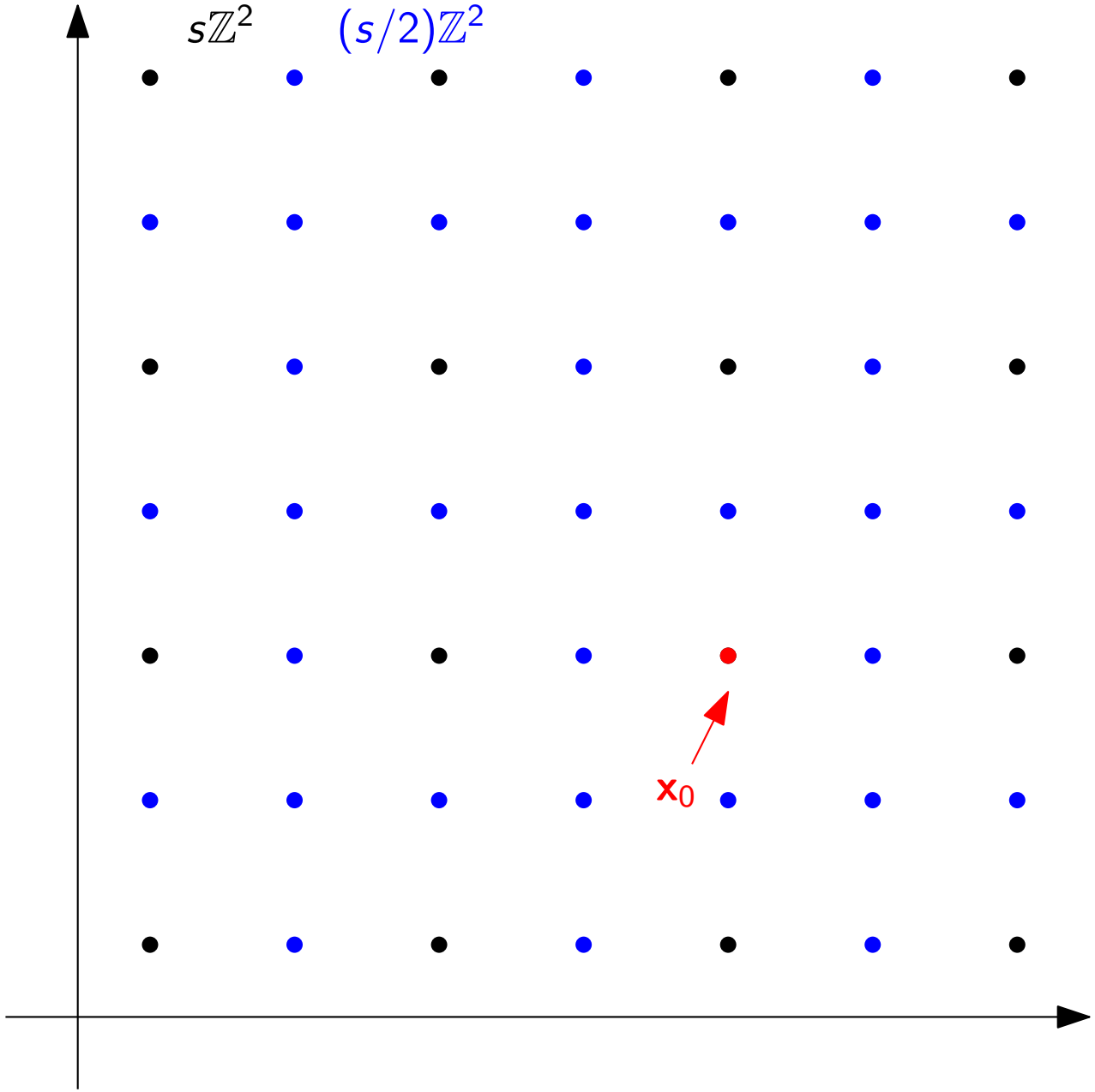
- [Cslovjecsek, Eisenbrand, Hunkenschröder, Rohwedder, Weismantel; SODA 2021, Proposition 4.1] shows instances of IP with
  - small $\mathrm{td}_D(A) + \|A\|_\infty$
  - but where integer optima are $\Omega(n)$ far in the ℓ_∞-norm from any continuous optimum.

Dual Algorithm: Reexamine the Scaling



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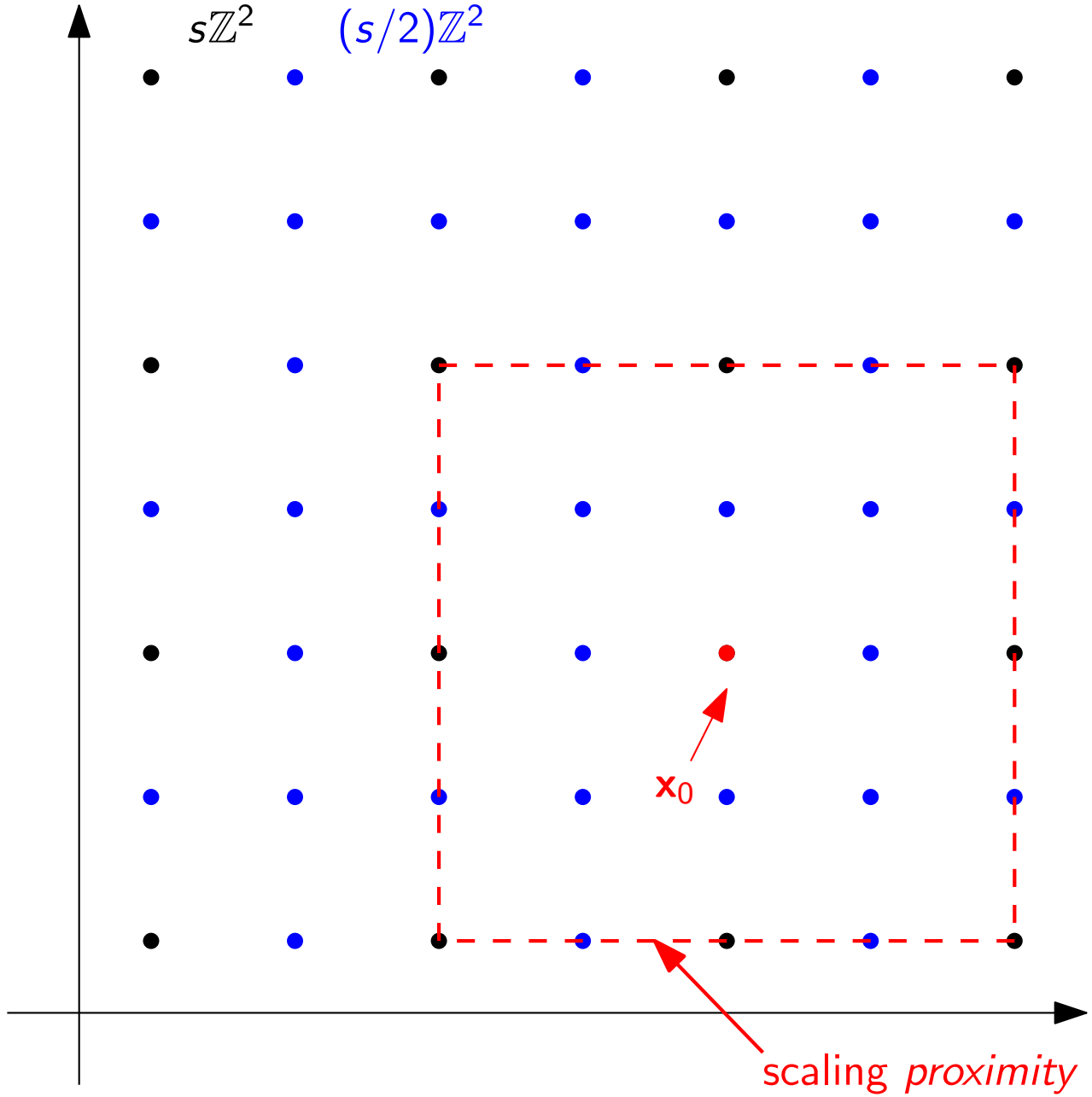

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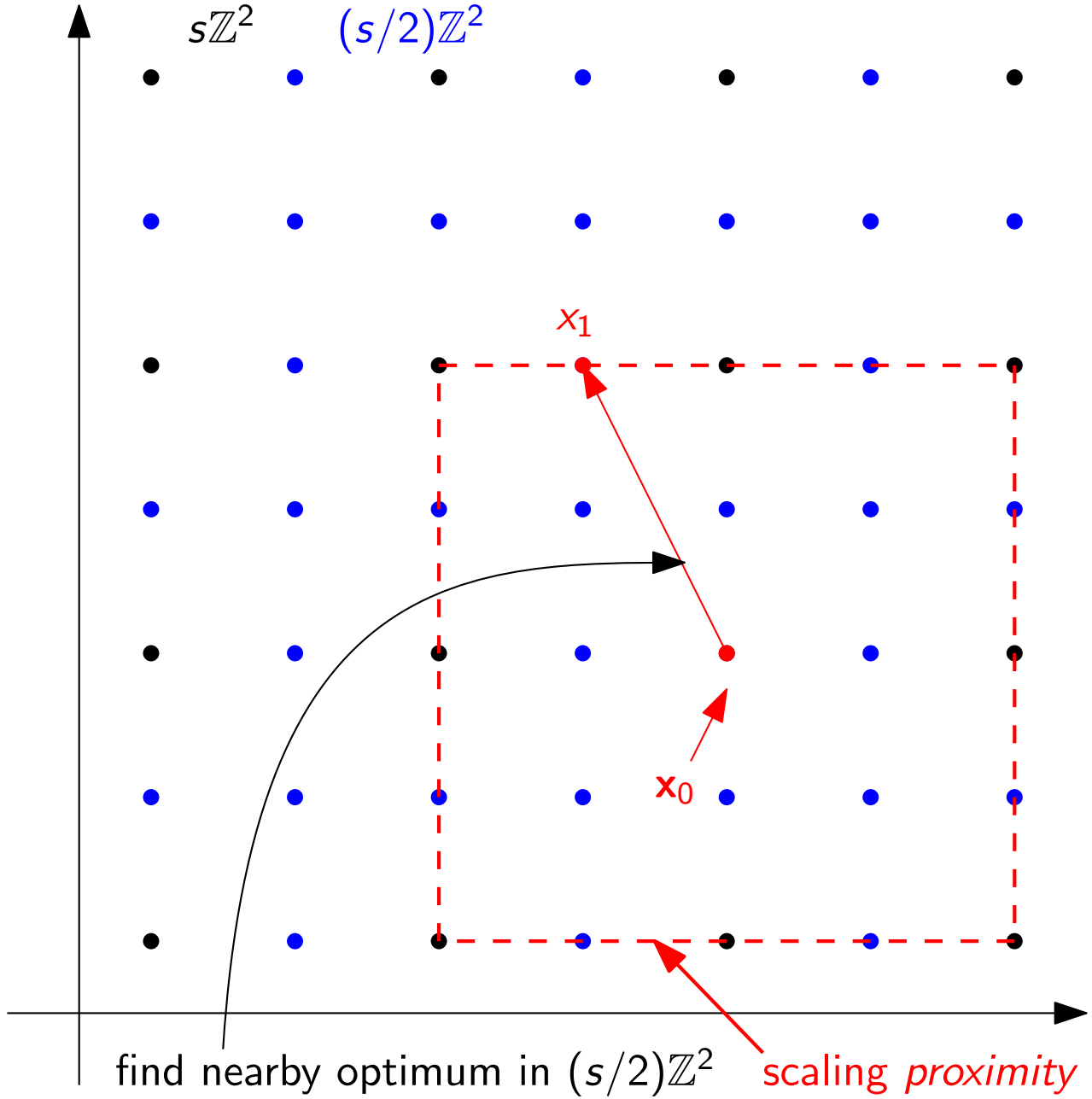

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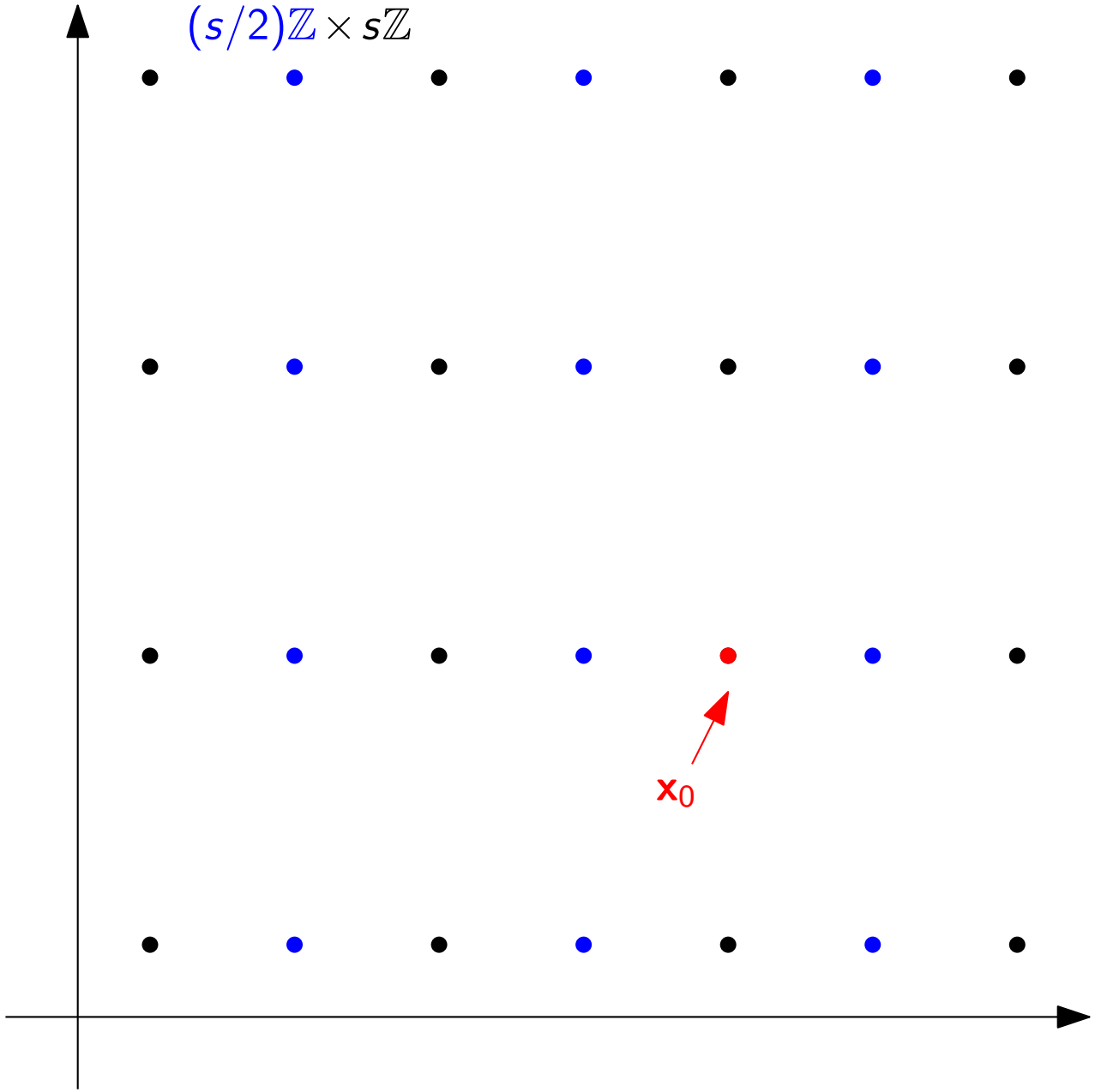

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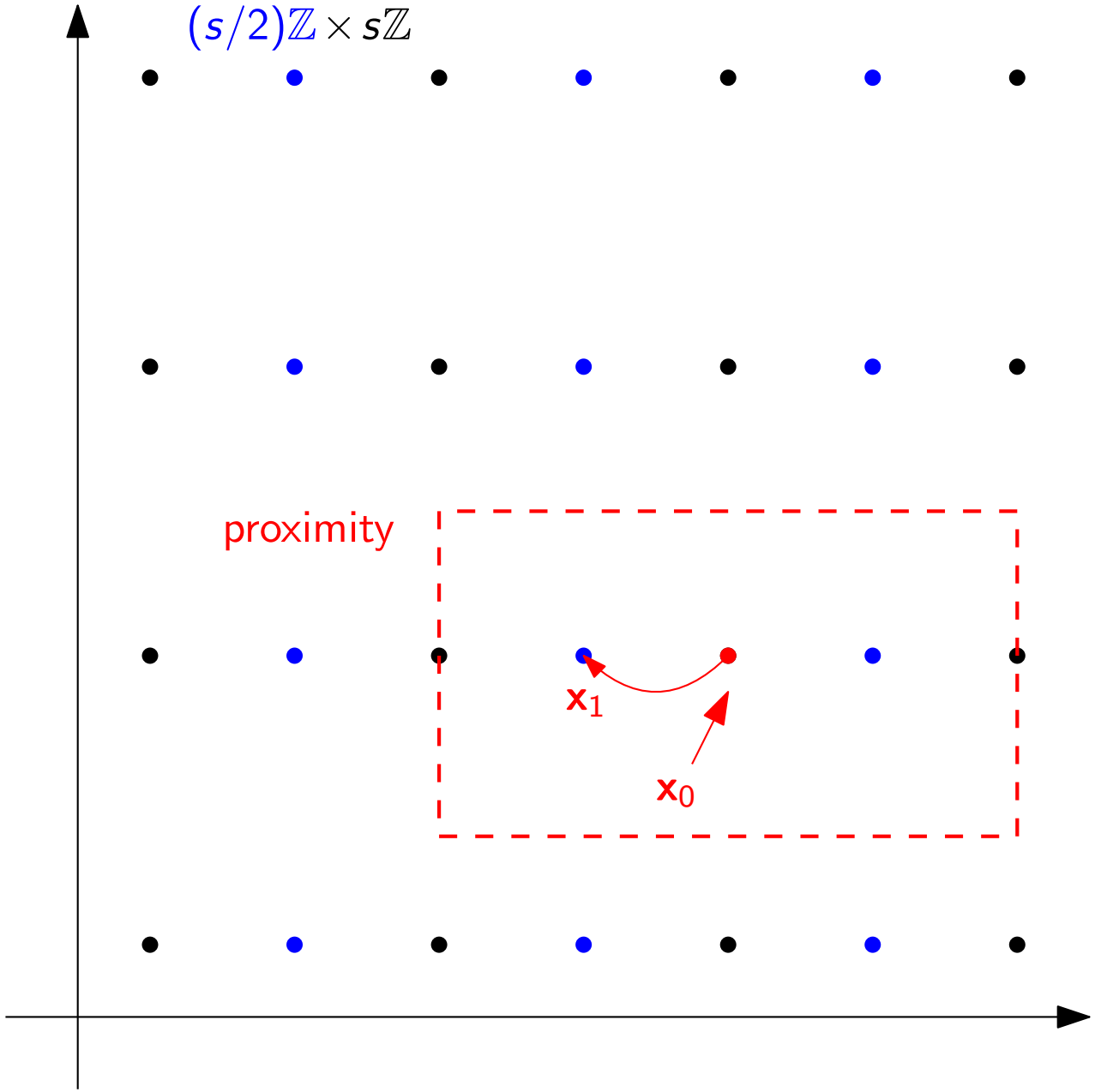

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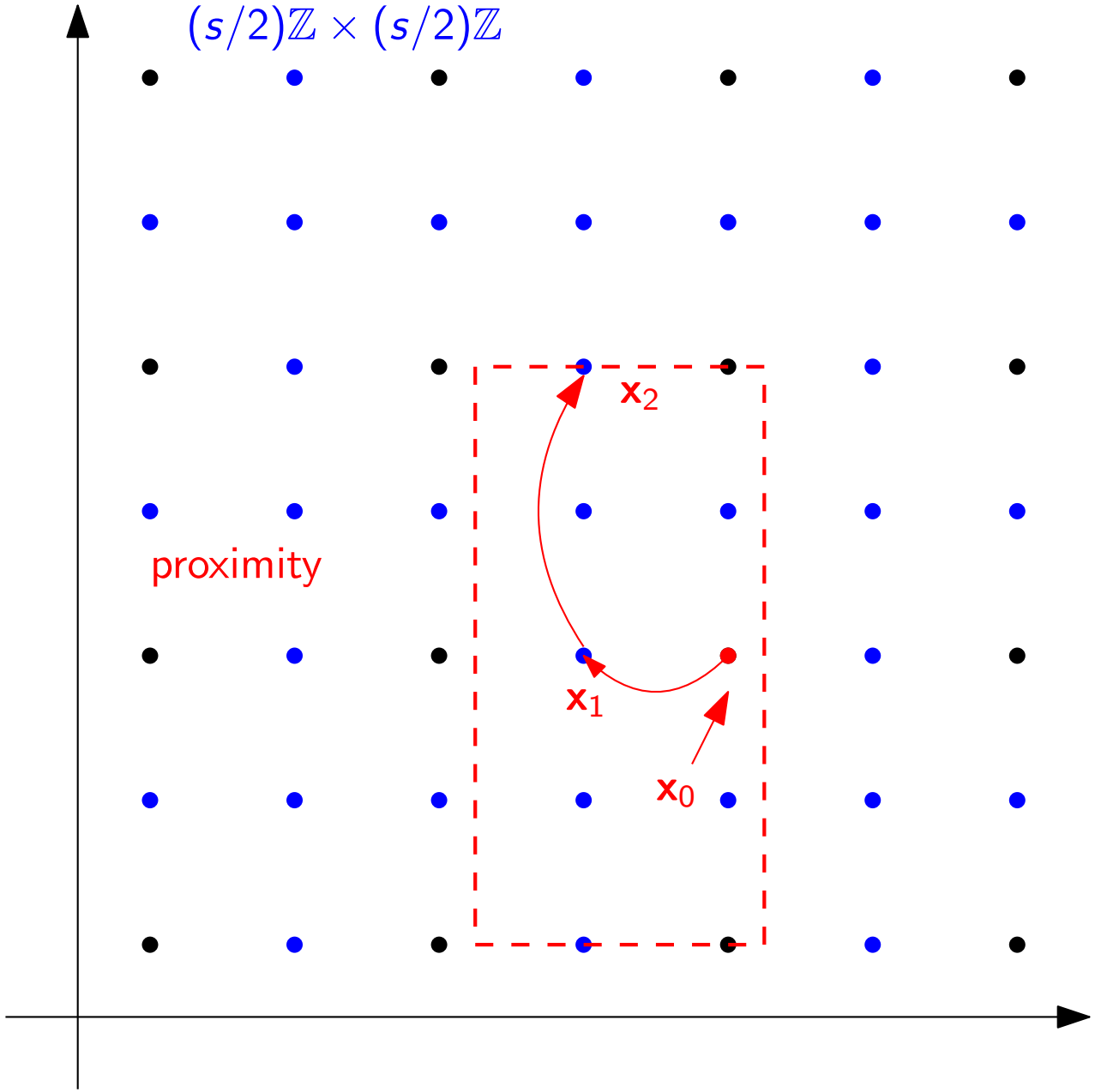

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- Overall dependence on n is $n \log n$.

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Thank you for your attention!

https://arxiv.org/abs/2505.22212

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