Generalized k-Center: Distinguishing Doubling and Highway Dimension

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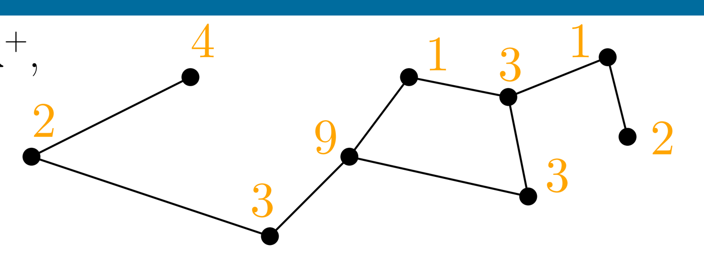
Capacitated k-Center (CkC)

Input

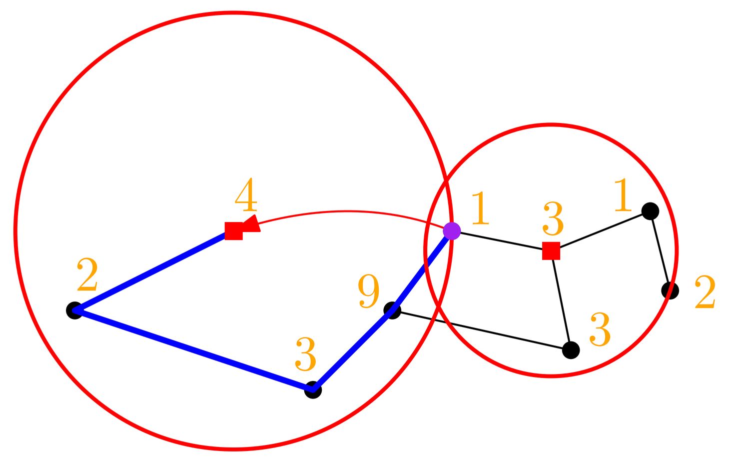
- graph $G = (V, E)$ with edge lengths $\ell \colon E \to \mathbb{R}^+$,
- integer k,
- *capacities* $c \colon V \to \mathbb{N}$.



Goal

Find $S \subseteq V$ and an *assignment* $\varphi \colon (V \setminus S) \to S$ such that

- $|S| \le k$,
- for every $u \in S$, $|\varphi^{-1}(u)| \le c(u)$, and
- $\max_{v \in V \setminus S} \mathrm{dist}(v, \varphi(v))$ is minimal.

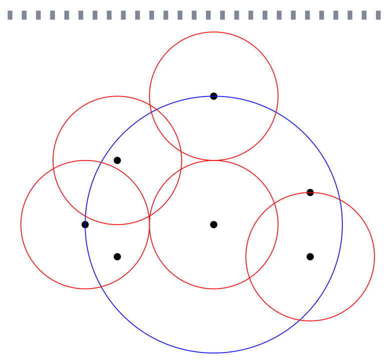


Doubling dimension (Δ)

...of graph $G = (V, E)$ is the smallest $\Delta \in \mathbb{N}$ such that

- the ball $B(u, r)$ for every $u \in V$ and every $r \in \mathbb{R}^+$
- is contained in $\cup_{v \in U} B(v, r/2)$ for some $U \subseteq V$ with $|U| \le 2^\Delta$.

$\rightsquigarrow$ d-dimensional ℓ_q metrics have doubling dimension $\mathcal{O}(d)$.



Overcoming lower bounds in special settings

	Doubling Dimension (Δ)	Highway dimension (h)
Capacitated k-Center	$k^k/\varepsilon^{\mathcal{O}(k\Delta)} \cdot \mathrm{poly}(n)$ Theorem 2	$\exists c > 1$: no c-approximation in $\mathcal{O}_\varepsilon(f(k,h) \cdot \mathrm{poly}(n))^{\dagger,\S}$ Theorem 1
k-Center	$k^k/\varepsilon^{\mathcal{O}(k\Delta)} \cdot \mathrm{poly}(n)$ Feldmann, Marx. 2020	$f(k,h,\varepsilon) \cdot \mathrm{poly}(n)^\dagger$ Becker, Klein, Saulpic. 2018
k-Median, k-Means, Facility Location	$2^{(1/\varepsilon)^{\mathcal{O}(\Delta^2)}} \cdot \mathrm{poly}(n)$ Cohen-Addad, Feldmann, Saulpic. 2021	$n^{(2h/\varepsilon)^{\mathcal{O}(1)}}$ Feldmann, Saulpic. 2021
TSP, Steiner Tree	$\exp\{2^{\mathcal{O}(\Delta)} \cdot (4\Delta \log n/\varepsilon)^\Delta\}$ Talwar. 2004	$\exp\left\{\mathrm{polylog}(n)^{\mathcal{O}(\log^2(h/\varepsilon))}\right\}$ Feldmann, Fung, Könemann, Post. 2018

†: f: computable function §: unless $\mathsf{FPT} = \mathsf{W}[1]$

Designing PTAS'es/EPAS'es for low highway dimension graphs

Usual approach:

1. Obtain an EPAS for low doubling dimension graphs.
2. Generalize the approach to low highway dimension graphs.

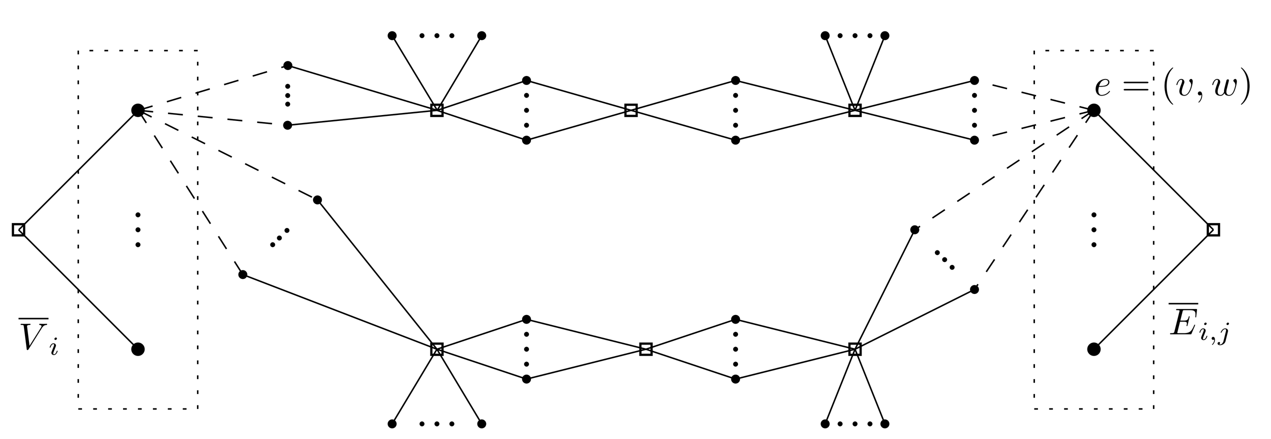
E.g., the last two rows of the table above.

Theorems 1 and 2 combined show a **first** example of a problem where this approach is not possible!

Hardness for highway dimension graphs (Theorem 1)

Approach

- Dom et al. (2008) show that CkC is $\mathsf{W}[1]$-hard for low treewidth graphs.
- We add edge weights to this reduction to obtain the hardness result for low highway dimension graphs.



Dashed edges have weight 1 and full edges have weight 8.

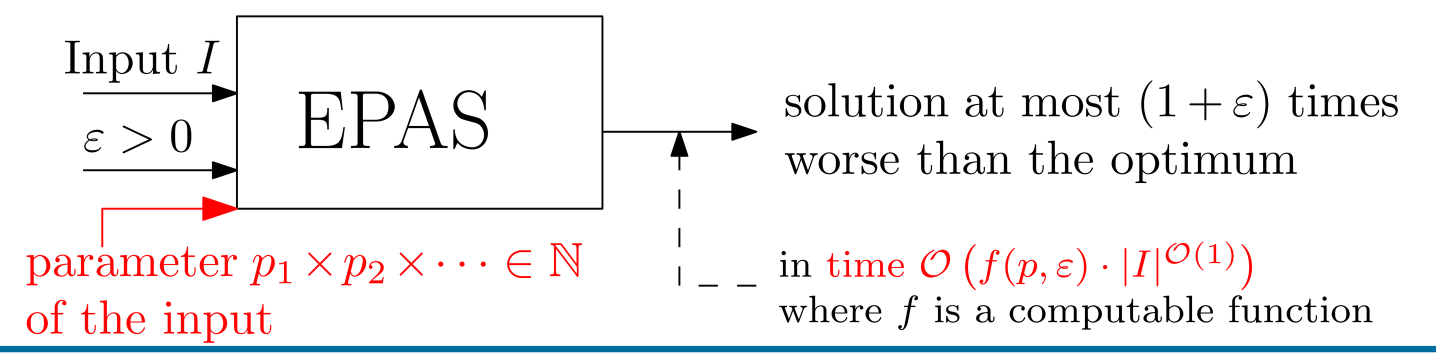
Polynomial algorithms: state of the art

- Cygan, Hajiaghayi, Khuller. 2012: CkC is NP-hard to even $(3-\varepsilon)$-approximate.
- An, Bhaskara, Chekuri, Gupta, Madan, Svensson. 2015: CkC can be 9-approximated.

Can we overcome this lower bound in special settings?

E.g., Euclidean spaces, real world, planar graphs,...

Efficient Parameterized Approximation Scheme



Highway dimension (h)

Shortest Path Cover

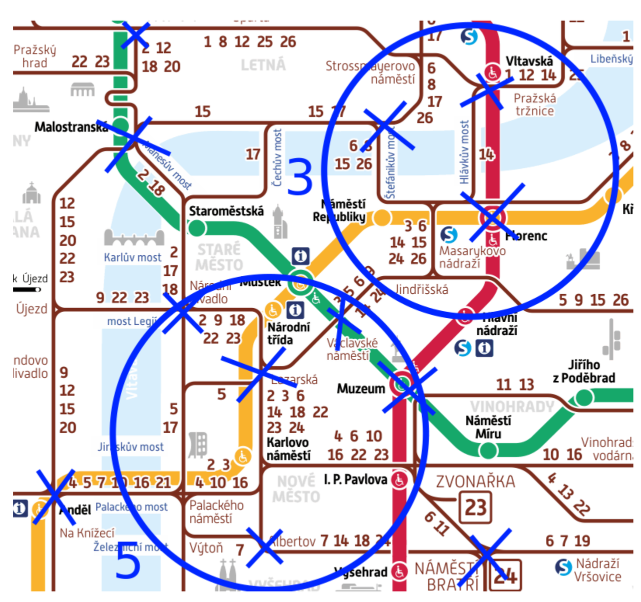
- G: edge-weighted graph. Fix a *scale* $r \in \mathbb{R}^+$.
- $\mathcal{P}_r$: set of paths of G such that
  - they are a shortest path between their end-points,
  - their length is more than r and at most $2r$.

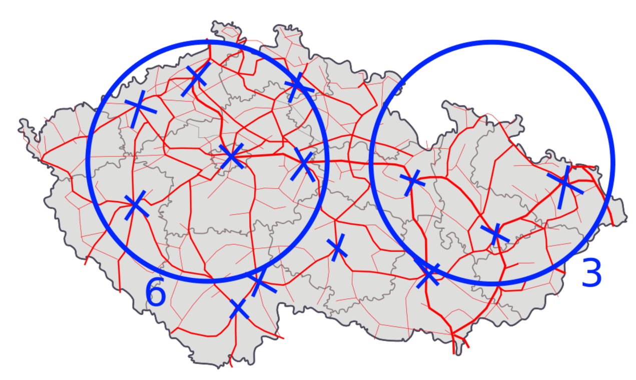
shortest path cover $\mathrm{SPC}_r(G)$: hitting set for $\mathcal{P}_r$.

Highway dimension

highway dimension of an edge-weighted graph G:

- smallest integer h such that,
- for any scale $r \in \mathbb{R}^+$,
- there exists $H := \mathrm{SPC}_r(G)$ so that,
- $|H \cap B(u, 2r)| \le h$ for every $u \in V(G)$.



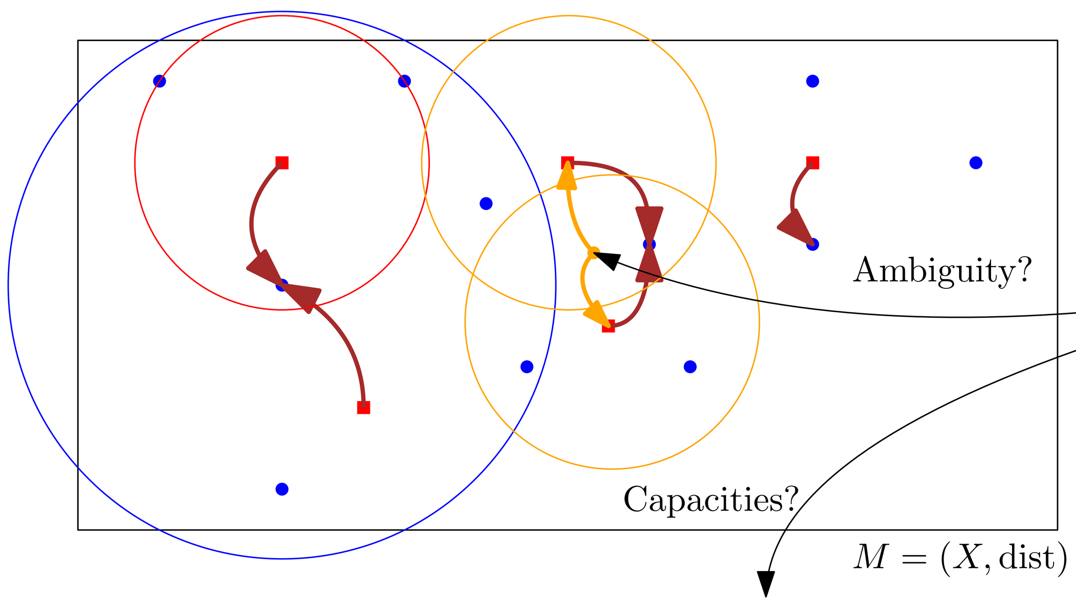


Is approximation necessary in these special settings?

- Feldmann and Marx, 2020: k-Center is $\mathsf{W}[1]$-hard in graphs of constant Δ for parameters k, h, and pathwidth. $\Rightarrow$ must approximate even in parameterized setting.
- Feder and Greene, 1988: k-Center is NP-hard to $(1.822-\varepsilon)$ or $(2-\varepsilon)$-approximate in two-dimensional Euclidean resp. Manhattan metric. $\Rightarrow$ cannot parameterize only by Δ.

(Capacitated) k-Center algorithm (Theorem 2)

For better intuition, view the input graph as a metric space $M = (X, \mathrm{dist})$ with dist induced by shortest-path distances.



- Optimum solution of cost OPT.
- *Net*: $Y \subseteq X$ such that
  - $\forall x \in X \exists y \in Y : d(x,y) \le \varepsilon\,\mathrm{OPT}$
  - $\forall y_1 \ne y_2 \in Y : d(y_1, y_2) > \varepsilon\,\mathrm{OPT}$

CkC obstacles

Solution verification?

- Replace every optimum center by its nearest net point.
  - $\Rightarrow$ We get a $(1+\varepsilon)$-approximate solution.
- It can be shown that $|Y| \le k(1/\varepsilon)^{\mathcal{O}(\Delta)}$.
  - $\Rightarrow$ Guess the k-tuple near the optimum centers to get an EPAS with parameters k, ε, and Δ.

Dealing with CkC obstacles (sketch)

Capacities. Replace every optimum center with a nearest net point with the highest capacity.

Ambiguity. View the k-tuple near the optimum center set as a multiset.

Solution verification. Reduce to network flows.