Generalized k-Center: Distinguishing Doubling and Highway Dimension

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Capacitated k-Center

Input

- graph $G = (V, E)$ with edge lengths $\ell \colon E \to \mathbb{R}^+$,
- integer k,
- *capacities* $c \colon V \to \mathbb{N}$.

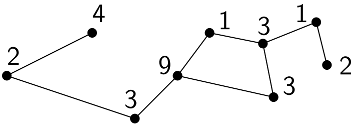

Figure: $\textsc{CkC}$ input with $k = 2$.

Capacitated k-Center: Goal

Find $S \subseteq V$ and an *assignment* $\varphi \colon (V \setminus S) \to S$ such that

- $|S| \leq k$,
- for every $u \in S$, $|\varphi^{-1}(u)| \leq c(u)$, and
- $\max_{v \in V \setminus S} \mathrm{dist}(v, \varphi(v))$ is minimal.

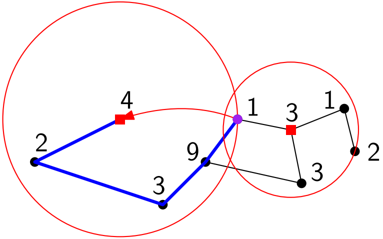

Figure: CKC solution for $k = 2$.

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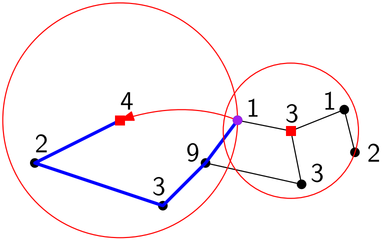

Figure: CKC solution for $k = 2$.

When $c(u) = |V|$ for every $u \in V \Rightarrow k$-CENTER.

Capacitated k-Center: Solution Prospects

CAPACITATED k-CENTER is NP-hard.

$\Rightarrow$ cannot solve exactly in polynomial time assuming $P \neq NP$.

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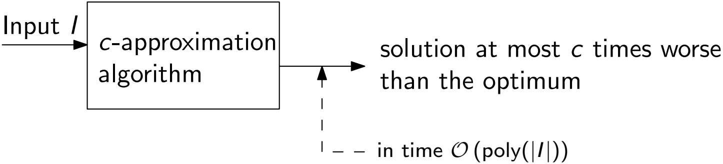
c-approximation algorithm



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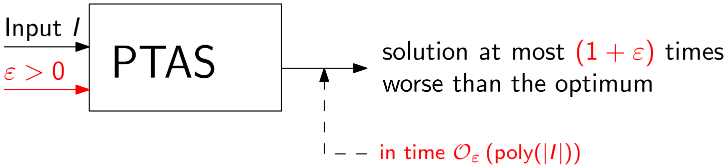
Polynomial-time approximation scheme



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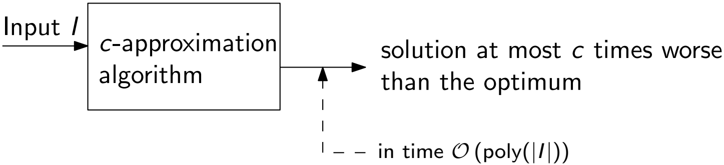

An, Bhaskara, Chekuri, Gupta, Madan, Svensson. 2015

There is a 9-approximation algorithm for $\textsc{CkC}$.

Cygan, Hajiaghayi, Khuller. 2012

There is no $(3-\varepsilon)$-approximation algorithm for $\textsc{CkC}$ unless $\mathrm{P}=\mathrm{NP}$.

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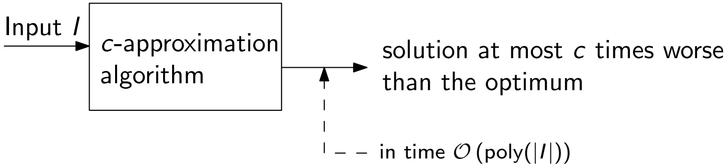

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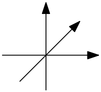
Question

Are there settings where we can overcome this lower bound?

Planar graphs, Euclidean spaces, real world, ...

Special Settings?



	Doubling Dimension (Δ)	
Capacitated k-Center	generalizes the dimension of ℓ_q spaces	
k-Center	$k^k/\varepsilon^{\mathcal{O}(k\Delta)} \cdot \mathrm{poly}(n)$ Feldmann, Marx. 2020	
k-Median, k-Means, Facility Location	$2^{(1/\varepsilon)^{\mathcal{O}(\Delta^2)}} \cdot \mathrm{poly}(n)$ Cohen-Addad, Feldmann, Saulpic. 2021	
TSP, Steiner Tree	$\exp\{2^{\mathcal{O}(\Delta)} \cdot (4\Delta \log n/\varepsilon)^{\Delta}\}$ Talwar. 2004	

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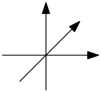

	Doubling Dimension (Δ)	Highway dimension (h)
CAPACITATED k-CENTER		captures properties of transportation networks
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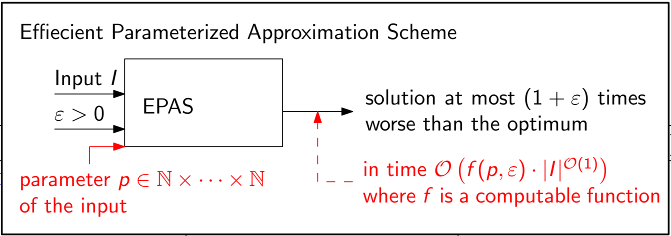

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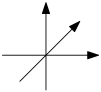

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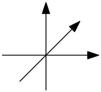

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Doubling Dimension

- Let $M = (X, \text{dist})$ be a metric space.

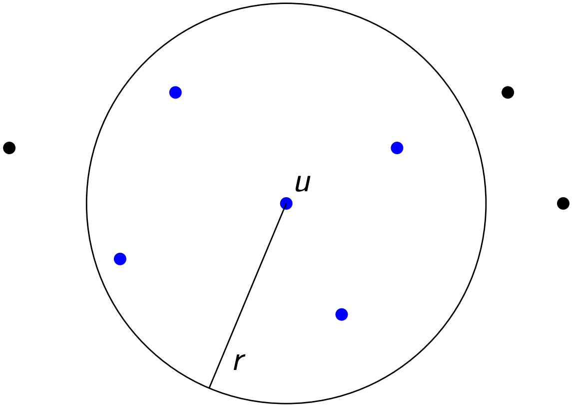
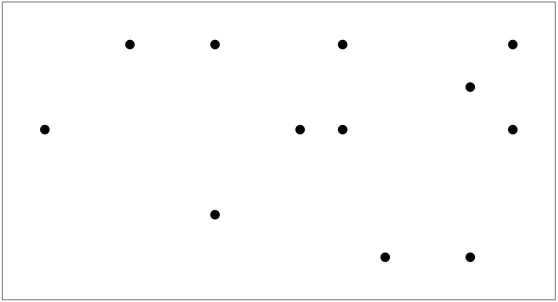


Figure: $B_r(u)$: Ball of radius r.

Doubling Dimension

Doubling dimension $\Delta(M)$: smallest $\Delta \in \mathbb{N}$ such that



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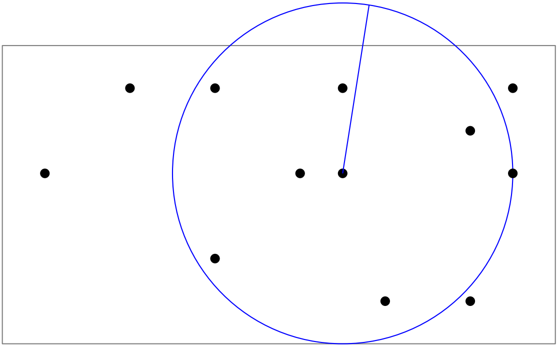
- the ball $B_r(u)$ for every $u \in X$ and every $r \in \mathbb{R}^+$



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- the ball $B_r(u)$ for every $u \in X$ and every $r \in \mathbb{R}^+$
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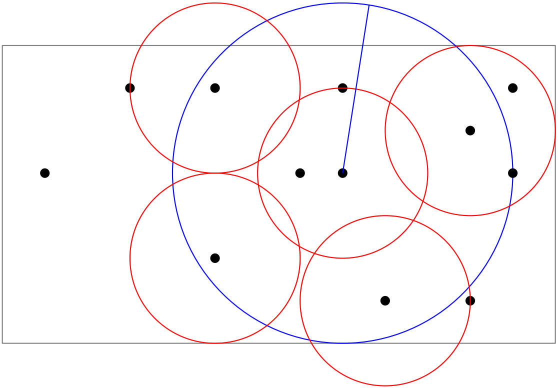

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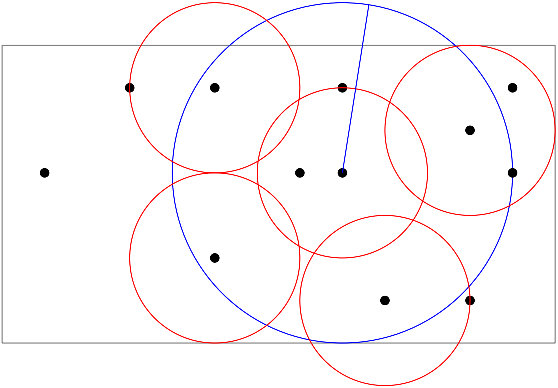
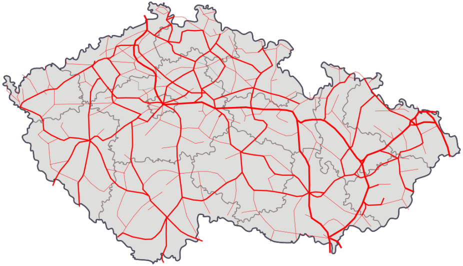


$\rightsquigarrow$ d-dimensional ℓ_q metrics have doubling dimension $\mathcal{O}(d)$.

Highway Dimension: Shortest Path Cover

- Let G be an edge-weighted graph and fix a *scale* $r \in \mathbb{R}^+$.
- Let $\mathcal{P}_r$ be the set of paths of G such that
  - they are a shortest path between their endpoints,
  - their length is more than r and at most $2r$.

(a) Metro and tram network in Prague city center.



(b) Czech railway network.

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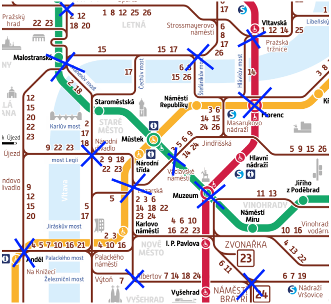
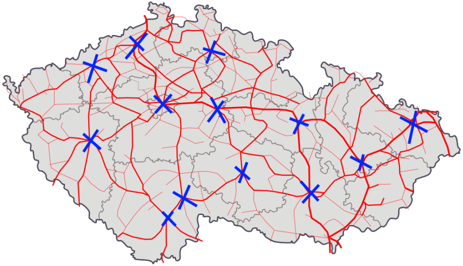




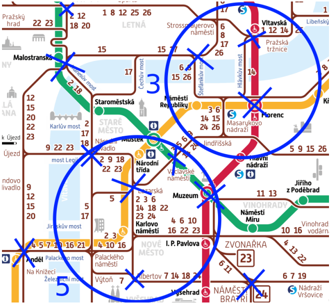
The *shortest path cover* $\mathrm{SPC}_r(G)$ is a hitting set[1] for $\mathcal{P}_r$.

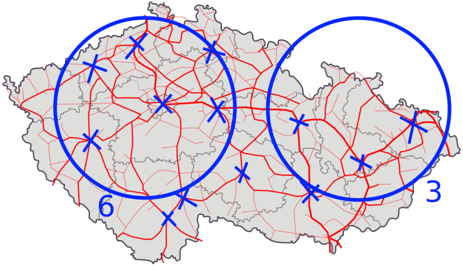
[1] For every $P \in \mathcal{P}_r$ we have $P \cap \mathrm{SPC}_r(G) \neq \emptyset$.

Highway Dimension

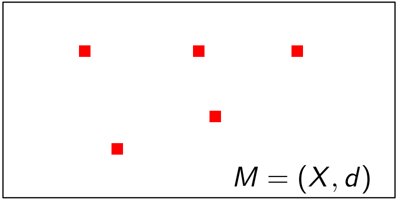
highway dimension of an edge-weighted graph G:

- smallest integer h such that,
- for any scale $r \in \mathbb{R}^+$,
- there exists $H := \mathrm{SPC}_r(G)$ so that,
- $|H \cap B_{2r}(u)| \leq$ h for every $u \in V(G)$.



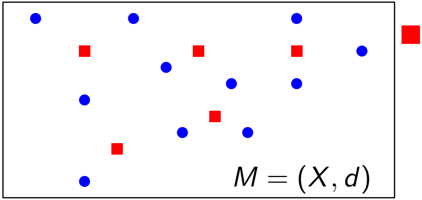


k-CENTER algorithm



Optimum solution of cost OPT.

k-Center algorithm



● *Net*: $Y \subseteq X$ such that

- $\forall x \in X \exists y \in Y : d(x, y) \leq \varepsilon\,\mathrm{OPT}$, and
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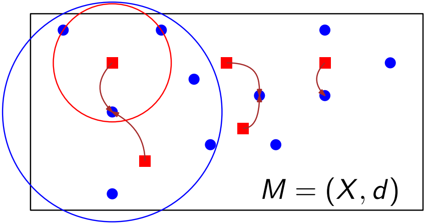

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$\Rightarrow$ We get a $(1 + \varepsilon)$-approximate solution.

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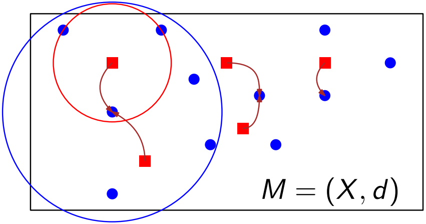

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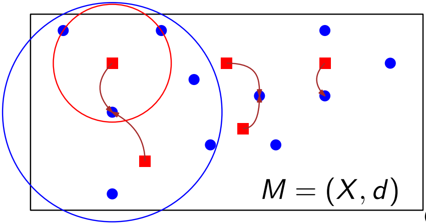
$\Rightarrow$ Guess the k-tuple near the optimum centers to get an EPAS with parameters k, ε, and Δ.

CKC algorithm obstacles



■ Optimum solution of cost OPT.

CKC obstacles

Capacities?

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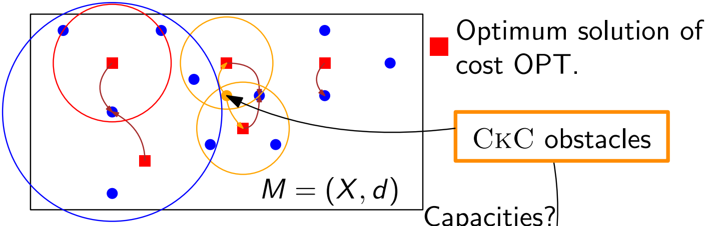

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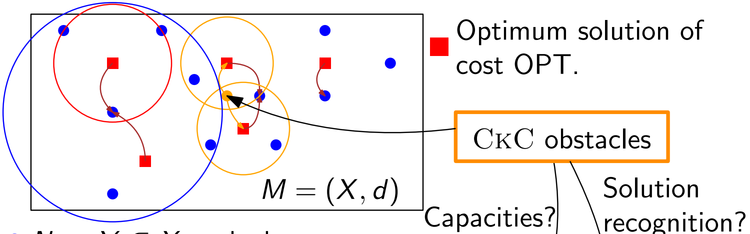

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Thank you for your attention!

Questions, comments, ...?