Exam from Mathematical Analysis II. (practice)

Time: 3 hours

It is not allowed to use any electronics or brought written/printed materials.

1. Riemann integral
   (a) *(3 points)* Define upper and lower Riemann integral.
   (b) *(6 points)* Describe properties of Riemann integral and its relation primitive function and to Newton integral.
   (c) *(4 points)* Give examples of functions that have only one type of integral (only Newton or only Riemann), if they exist.
   (d) *(6 points)* Which functions are Riemann integrable and which are not? (Preferably give classes of functions, not just individual examples.)
   (e) *(5 points)* Compute $\int_0^{9\pi/2} \sin^n x \cos x \, dx$.
   (f) *(6 points)* Using integrability criterion, prove that if f is Riemann integrable on $[a, b]$, $|f|$ is integrable on $[a, b]$ as well.

2. Extrema of multivariate functions
   (a) *(3 points)* Define extrema of multivariate function.
   (b) *(6 points)* A function f of two variables satisfies $\frac{\partial f}{\partial x}(0,0) = c$ and $\frac{\partial f}{\partial y}(0,0)$ not exist. For case $c = 0$ and $c = 1$ either argue that f cannot have a local extreme in $(0,0)$ or give an example of a function f (with prescribed partial derivatives) that has a local extreme in $(0,0)$.
   (c) *(6 points)* Find all the extrema of the function $f(x, y) = 2x^3 + 4y^2 - 2y^4 - 6x$ and determine whether they are maxima, minima or saddle points.
   (d) *(5 points)* State Lagrange multipliers theorem.

3. Metric spaces
   (a) *(3 points)* Define a compact set in a metric space.
   (b) *(10 points)* Prove that a compact set is closed and bounded. Is it true the other way round?
   (c) *(6 points)* Prove that (M, d) is a metric space if M is a nonempty set, f is a function from M to positive reals and d is defined as

$$d(x,y) = \begin{cases} 0 & x = y \\ f(x) + f(y) & x \neq y. \end{cases}$$

   (d) *(6 points)* Characterize convergent sequences in this space.