8th problem set for Probability and Statistics — April 9th

Recall that the cumulative distribution function F_X is defined by

$$F_X(x) = P(X \le x).$$

If X is continuous, then

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

for a non-negative function f_X (the density of X). Then

$$P(X \in A) = \int_A f_X(t)dt, \qquad \text{thus} \qquad P(a \le X \le b) = \int_a^b f_X(t)dt$$

Also, $\mathbb{E}(X) = \int_{-\infty}^{\infty} x\ f_X(x)dx$ and in general

$$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(t)\ f_X(t)dt.$$

Just as for discrete random variables, here also holds that $\text{var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$.

Before solving the problems, you might want to recall how to compute definite integrals using primitive functions.

Using F and f

1. For a random variable X with the distribution function F_X, express
(a) $P(X \in (0, 1])$ (b) $P(X > 0)$ (c) * $P(X < 0)$ (d) * $P(X \in [0, 1])$

2. Solve the previous part again, now for a random variable X with density f_X.

3. Let X be a random variable satisfying $P(X = x) = 0$ for every x. (Actually, there is nothing strange about this, and in fact it happens for every continuous random variable.)
Express the distribution function of the following random variables using F_X
(a) $-X$. (b) $X^+ = \max(0, X)$, (c) $|X|$.

4. Let X be a random variable with density $f_X(t) = 1/t^2$ for $t \ge 1$ and $f_X(t) = 0$ otherwise.
(a) Verify that this is a density function.
(b) Determine $\mathbb{E}(X)$.
(c) Compute the distribution function F_X.
(d) Determine $P(2 \le X \le 3)$.
(e) Let $Y = 1/X$. What is the distribution function of the random variable Y?
(f) Determine the density of the random variable Y.

5. We say that X has an exponential distribution, $X \sim Exp(\lambda)$, if

$$F_X(x) = 1 - e^{-\lambda x} \quad \text{for } x \ge 0\text{, otherwise } 0.$$

Find f_X. We will show in the lecture that $\mathbb{E}(X) = 1/\lambda$.

6. Let's assume that at a post office counter, the time for serving one customer follows an exponential distribution with an average of 4 minutes.
(a) What is the parameter λ, what is the distribution function?
(b) What is the probability that we will wait more than 4 minutes?
(c) What is the probability that we will wait between 3 and 5 minutes?

More practice problems

7. The average lifespan of a hard disk is 4 years. Let's assume that this time is described by a random variable with an exponential distribution. (This is not a realistic assumption, see e.g., `https://www.backblaze.com/blog/how-long-do-disk-drives-last/`.)

(a) What is the probability that the disk will fail within the first three years?
(b) What is the probability that it will last at least 10 years?
(c) After what time will 10% of the disks fail?

8. Plutonium-238 has a half-life of 87.7 years. We will model its decay using an exponential distribution: for each atom, we consider the time until decay as an independent random variable with the distribution $Exp(\lambda)$.

(a) What is λ?
(b) What is the average lifespan of a plutonium-238 atom?
(c) After how much time will 90% of the atoms decay?
(d) What percentage of atoms will decay after 50 years? (Some cardiac pacemakers use plutonium-238 as an energy source. `https://en.wikipedia.org/wiki/Plutonium-238#Nuclear_powered_pacemakers`)

9. The time until we see a meteor is exponentially distributed with a mean of 1 minute.

(a) What is the probability that we will have to wait more than 5 minutes?
(b) What is the probability that we will see it within at most one minute?
(c) * What is the distribution of the time when we see the second meteor? The third, ... (We assume that individual meteors are independent.)

8th homework, due next week

Let F_X be given by the formula $F_X(x) = x/3$ for $x \in [0, 3]$, $F_X(x) = 0$ for $x < 0$, and $F_X(x) = 1$ for $x > 3$. Let $Y = 1/X$ and $Z = X^2$. Compute

(d) $P(1 \leq X \leq 2)$ (e) $P(X \leq Y)$ (f) $P(X \leq Z)$ (g) the density function f_X. (h) the distribution functions F_Y and F_Z.