Exercise Sheet 7 from Probability and Statistics — April 2, 2024

Recognizing Random Variables

1. The probability of a data breach at our server for each given day is 0.01, independently for each day. Let T be the number of days until the first data breach. What is the distribution of T, $\mathbb{E}(T)$, $\mathrm{var}(T)$? What is the probability that the server remains secure for an entire year?

2. Each software test can either find a bug (this we caount as success) or not (this we count as failure). Assume the probability of finding a bug in one test is 0.05, and a developer performs 20 independent tests, denote X as the number of bugs found. What is the distribution of X, $\mathbb{E}(X)$, $\mathrm{var}(X)$? What is the probability of finding exactly three bugs?

3. Historical data show that our server receives an average of 30 requests per minute. Use the Poisson distribution to determine the probability that the server will receive exactly 40 requests in the next minute.

Variance

Reminder:

- definition $\mathrm{var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2)$
- theorem $\mathrm{var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$
- derived quantities: standard deviation $\sigma_X = \sqrt{\mathrm{var}(X)}$
- coefficient of variation $CV_X = \sigma_X / \mathbb{E}(X)$ (if $\mathbb{E}(X) > 0$)

4. Assume that solving one problem takes X minutes, where $X = 1, 2, \dots,$ or 5. The duration is random (dependent on weather), and the probability function is $p_X(1) = p_X(2) = 0.1$, $p_X(3) = p_X(4) = 0.2$, $p_X(5) = 0.4$. From the past, we know that $\mathbb{E}(X) = 3.7$. Find $\mathrm{var}(X)$ and σ_X.

5. By investing in a particular stock, a person can make a profit in one year of \$4000 with probability 0.3 or take a loss of \$1000 with probability 0.7. What is this person's expected gain? What is the variance?

6. For random variables X, Y, we have $Y = aX + b$ (where a, b are real numbers).
(a) Express $\mathrm{var}(Y)$ in terms of $\mathrm{var}(X)$.
(b) Repeat for σ_X and CV_X.

7. Let $X \sim Bin(100, 0.5)$ and $Y \sim 10Bin(100, .05)$ (thus, $Y/10$ has a binomial distribution $Bin(100, .05)$). Compute (using formulas from the lecture) $\mathbb{E}(X)$, $\mathrm{var}(X)$, σ_X, CV_X and the same for Y.

Bonus

8. Do the following *alternative definitions of variance* make sense? That is, do you learn anything interesting about random variable X when you find
(a) $\mathbb{E}(X - \mathbb{E}(X))$?
(b) $\mathbb{E}(|X - \mathbb{E}(X)|)$?
(c) $\mathbb{E}((X - \mathbb{E}(X))^k)$ for $k = 3, 4, \dots$.

9. Assume that two random variables (X, Y) are uniformly distributed on a circle with radius a. Then the joint probability density function is:

$$f(x,y) = \begin{cases} \frac{1}{\pi a^2} & , x^2 + y^2 = a^2 \\ 0 & , elsewhere \end{cases}$$

Find the expected value of X.

7th homework, due next week

10. Alice and Bob went to a dinner party for married couples. There were total 5 married couples in the party including Alice and Bob (i.e., 10 people). There is a big round table with 10 seats and each person is given a uniformly randomly chosen seat around the table.
(a) What is the probability that Alice is seated next to Bob?
(b) What is the expected number of couples that are seated next to each other? If X is the random variable denoting the number of couples that are seated next to each other, then What is the variance of X?