2nd problem set for Probability and Statistics — February 26–March 1

1. Prove that if A, B are events, we have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Conditional probability, Bayes theorem

2. What is the relationship between the statements $P(A \mid B) > P(A)$ and $P(B \mid A) > P(B)$?

3. In a box of 100 tangerines, there are four bad ones. Let's take out three tangerines one at a time (without returning them to the box). Let A_i denote the event „i-this tangerine is not spoiled".

(a) Calculate $P(A_1 \cap A_2)$. **Use conditional probability.**

(b) Calculate $P(A_1 \cap A_2 \cap A_3)$. Again, use conditional probability. If you don't know how to solve this part, find a useful formula at the end of second page.

4. We have three normal 6-sided dice and one die with three 1s and three 2s. We pick one of the dice uniformly at random and roll it.

(a) What is the probability of rolling a 1?

(b) If the outcome was 1, what is the probability that we picked a normal die?

5. Peter gets a lot of emails, but 80 % of them are spam. His spam filter correctly flags 90 % of the spam, but it also flags 5 % of the regular emails as spam.

(a) What percentage of emails will be marked as spam?

(b) What percentage of proper emails are among those marked as spam?

(c) What percentage of spam emails are among the emails that pass the test?

6. (Monty Hall Paradox) In a TV competition, the contestants (i.e., us) stand on a stage in front of three doors. Behind one door there is a car (that's what we want), behind each of the other two doors there is a goat (we don't want that). We choose one door and once we open it, we can take whatever is behind it. However, before we open it, the presenter opens one of the other doors, shows the goat behind it, and offers us the opportunity to change our choice. Should we do that? Will it increase the probability of getting a car? Note that the assignment has (at least) the following two variants:

(a) The presenter knows where the car is, and will always pick a door with a goat to open;

(b) The presenter tosses a coin to decide which door to open (of the two that we haven't chosen). If he had revealed the car, we would have lost, but that didn't happen.

To make it easier to talk about this problem: we pick door number 1, the car is behind a random door. After the moderator opens door 2 or 3, we change our choice. Calculate the probability that we win the car, in variants (a), (b).

7. Alice has n coins, Bob has $n+1$. They both flip all their coins and count how many times they get heads. Prove that the probability that Bob gets more heads than Alice is equal to 1/2. (Hint: Bob will put one coin aside and count all the others, and only then he will consider the last one.)

8. A variant of the envelope problem: In two envelopes, each envelope contains an amount given by some real number, each envelope contains a different real number. We are allowed to open one envelope and then decide whether to keep that one, or switch to the other one. With probability $> 1/2$, how can we get the envelope with the higher amount?

(A recipe for a solution: it won't be much more than 1/2, plus the probability depends on how much the two amounts differ. Use some increasing function $F : \mathbb{R} \to (0, 1)$. If x is the amount in the envelope that you picked, keep the envelope with probability $F(x)$.)

Bonus problems

9. (Simpson's Paradox) In this problem, we will have two kinds of candy: tasty yellow candies and disgusting green candies. However, we pick the candies without looking (or we are colorblind). The candies are in four containers: a white jar and a black jar and a red bag and a black bag. Decide whether the following strange phenomenon can happen:

- The likelihood or getting a tasty candy is greater if we take it out of the the red jar than if we take it out of the black jar.

- The likelihood or getting a tasty candy is greater if we take it out of the red bag than if we take it out of the black bag.
- Now we transfer all candies from red bag into the red jar, and all candies from the black bag into the black jar. After this operation, we are more likely to pull a tasty candy if we reach into the black jar than if we reach into the red jar.

10. (Prosecutor's fallacy) Mrs. C's two children died shortly after birth. She's charged with double murder. The prosecutor argues as follows: The probability of sudden infant death syndrome is 1/8500. So the probability that sudden infant death syndrome happens twice in a row is $1/8500^2$. Hence the probability that Mrs. C is innocent is $1/8500^2$, which is very low.

Formulate the prosecutor's arguments in the language of probability and find two errors in them.

More practice problems

11. We flip a coin twice. Are we more likely to get two heads assuming the first flip was heads, OR two heads assuming that at least one of the two flips was heads?

12. We have k containers, each containing a white and b black balls. We pick a random ball from the first, throw it into the second. Then we pick a random ball from it, throw it into the third, etc. What is the probability of getting a white ball out of the last jar?

13. There are m white and n black balls in a box. Two players take turns taking balls out of the box, the first one to take out a white ball loses. What is the probability $p(m, n)$ that the first player loses? (Construct a recurrence formula, i.e., a formula for $p(m, n)$ using $p(m', n')$ for $m' \leq m$, $n' \leq n$.)

14. There are a black and b white balls in the urn. We draw balls from it one by one (without returning them). What is the probability that the first ball drawn is black? Second, third, ...?

15. The logical formula $A \Rightarrow B$ is equivalent to $\neg B \Rightarrow \neg A$. We'll look at analogies involving probability.
(a) Show that if $P(B \mid A) = 1$, then $P(A^c \mid B^c) = 1$.
(b) Show, however, that it is possible for $P(B \mid A) \doteq 1$ but $P(A^c \mid B^c) \doteq 0$.

Useful formulas:

- If $A_1, \ldots, A_n$ are events and $P(A_1 \cap \cdots \cap A_n) > 0$, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2) \ldots P(A_n \mid \bigcap_{i=1}^{n-1} A_i)$$

- If $B_1, B_2, \ldots$ is a partition of Ω, and A is an event, then

$$P(A) = \sum_i P(A \mid B_i)P(B_i)$$

(Whenever $P(B_i) = 0$, then technically $P(A \mid B_i)$ is undefined. Let us consider these summands to be equal to 0.)

- Under the assumptions of the previous section,

$$P(B_j \mid A) = \frac{P(B_j)P(A \mid B_j)}{\sum_i P(B_i)P(A \mid B_i)}$$

2nd homework, due next week

We have a hat with 4 white, 4 black and 2 red balls. We will pull out 3 balls, one after another (without putting them back). What is the probability that the third of these balls is red?