Stationary distributions of finite Markov chains

Recall that a finite Markov chain with n states is represented by a non-negative matrix $P \in \mathbb{R}^{n \times n}$ with column sums equal to 1. We represent a distribution on the states by a non-negative vector $\mathbf{p}$ with the sum of entries equal to 1. If $\mathbf{p}$ is a distribution on the n states, then $P\mathbf{p}$ is the distribution after a step of the Markov chain and $P^k\mathbf{p}$ is the state after k steps.

Definition 1. A distribution $\boldsymbol{\pi} \in \mathbb{R}^n$ is called a stationary distribution of a Markov chain P if $P\boldsymbol{\pi} = \boldsymbol{\pi}$.

Theorem 2. *Suppose that P is an irreducible and aperiodic Markov chain. Then there exists a unique stationary distribution $\boldsymbol{\pi}$. Moreover, for every i, j, $\lim_{t\to\infty} P^t_{i,j}$ exists and is equal to π_i.*

Lemma 3. *For P aperiodic and irreducible, there exist $k \geq 1$ and $\delta > 0$ such that for all i, j, $P^k_{i,j} > \delta$.*

Proof. For a given i, let A be the set of all $t \geq 1$ such that $P^t_{i,i} > 0$. The set A is noempty as P is irreducible and it is closed under addition as $P^{t+t'}_{i,i} \geq P^t_{i,i} P^{t'}_{i,i}$. Furthermore, the aperiodicity of P implies that $gcd(A) = 1$, where $gcd(A)$ is the createst common divisor of (all the numbers in) A.

An easy number-theoretic fact is that such an A contains all but finitely many natural numbers. Let $a \in A$ be arbitrary. Since $gcd(A) = 1$, we can express 1 as a linear combination of elements of A with integer coefficients (some may be negative). This implies that for some $\alpha \in \mathbb{N}$, we can express $\alpha a + 1$ as a linear combination of elements of a with non-negative integer coefficients, simply by adding term $a \cdot b$ for all $b \in A$ with a negative coefficient sufficiently many times. Now for any $\beta \geq \alpha a$ and $\gamma = 0, \dots, a-1$, we can express $\beta a + \gamma$ as γ times the combination for $\alpha a + 1$ plus some multiple of a. Thus for any $t \geq \beta a$, we have $t \in A$.

Using the fact above for all i, we can find t_0 such that for all i and all $t \geq t_0$, $P^t_{i,i} > 0$. Now we claim that for $k = t_0 + n$, for all i, j, $P^k_{i,j} > 0$. Since P is irreducible, there exist $\ell \leq n$ such that $P^\ell_{i,j} > 0$: consider the shortest path in the underlying graph from state j to i. Then $P^k_{i,j} \geq P^{k-\ell}_{i,i} P^\ell_{i,j} > 0$ as $k - \ell \geq t_0$.

Finally, if for $P^k_{i,j} > 0$ for all i, j, then there also exists $\delta > 0$ such that $P^k_{i,j} > 0$ for all i, j, as the matrix has finitely many elements. □

We will work with the l_1-norm of vectors, which is just a sum of the absolute values of the entries, i.e., $||\mathbf{x}||_1 = \sum_{i=1}^n |x_i|$. We will also use the standard decomposition of a vector $\mathbf{x}$ to its positive and negative parts $\mathbf{x}^+$ and $\mathbf{x}^-$ defined by $x_i^+ = \max\{x_i, 0\}$ and $x_i^- = \max\{-x_i, 0\}$. Then $\mathbf{x} = \mathbf{x}^+ - \mathbf{x}^-$, $\mathbf{x}^+$ and $\mathbf{x}^-$ have disjoint support and $||\mathbf{x}||_1 = ||\mathbf{x}^+||_1 + ||\mathbf{x}^-||_1$.

Lemma 4. *Suppose that vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are non-negative and moreover all entries are at least α. Then $||\mathbf{v} - \mathbf{w}||_1 \leq ||\mathbf{v}||_1 + ||\mathbf{w}||_1 - 2\alpha n$.*

Proof. Direct calculation using the fact that for each coordinate $|v_i - w_i| = \max\{v_i, w_i\} - \min\{v_i, w_i\} = v_i + w_i - 2\min\{v_i, w_i\} \leq v_i + w_i - 2\alpha$. □

Lemma 5. *Let $D \in \mathbb{R}^{n\times n}$ be a matrix with all entries at least δ for some $\delta > 0$ and column sums equal to 1. Let $\mathbf{x}$ be decomposed to $\mathbf{x} = \mathbf{x}^+ - \mathbf{x}^-$ as described above. Then*
(i) $||D\mathbf{x}||_1 \le ||\mathbf{x}||_1 - 2\delta n \min\{||\mathbf{x}^+||_1, ||\mathbf{x}^-||_1\}$.
(ii) If $\mathbf{x}$ has both positive and negative entries, then $D\mathbf{x} \neq \mathbf{x}$.
(iii) If $\sum_{i=1}^n x_i = 0$ then $||D\mathbf{x}||_1 \le (1-\delta n)||\mathbf{x}||_1$.

Proof. (i): Consider the vectors $\mathbf{v} = D\mathbf{x}^+$ and $\mathbf{w} = D\mathbf{x}^-$; both $\mathbf{v}$ and $\mathbf{w}$ are non-negative as D, $\mathbf{x}^+$, and $\mathbf{x}^-$ are all non-negative. Since D has the column sums equal to 1 and $\mathbf{x}^+$, $\mathbf{x}^-$ are non-negative, $||\mathbf{v}||_1 = ||\mathbf{x}^+||_1$ and $||\mathbf{w}||_1 = ||\mathbf{x}^-||_1$, thus $||\mathbf{v}||_1 + ||\mathbf{w}||_1 = ||\mathbf{x}||_1$. Since D has all the entries at least $\delta > 0$ and $x_j^+ \ge 0$, we have $v_i \ge \delta \sum_{i=1}^n x_i^+ = \delta||\mathbf{x}^+||_1$. Similarly $w_i \ge \delta \sum_{i=1}^n x_i^- = \delta||\mathbf{x}^-||_1$. Now we use Lemma 4 with $\alpha = \delta \min\{||\mathbf{x}^+||_1, ||\mathbf{x}^-||_1\}$ to conclude that

$$\begin{aligned} ||D\mathbf{x}||_1 &= ||\mathbf{v} - \mathbf{w}||_1 \\ &\le ||\mathbf{v}||_1 + ||\mathbf{w}||_1 - 2\delta n \min\{||\mathbf{x}^+||_1, ||\mathbf{x}^-||_1\} = ||\mathbf{x}||_1 - 2\delta n \min\{||\mathbf{x}^+||_1, ||\mathbf{x}^-||_1\}. \end{aligned}$$

(ii): The assumption implies $\min\{||\mathbf{x}^+||_1, ||\mathbf{x}^-||_1\} > 0$. Then (i) implies $||D\mathbf{x}||_1 < ||\mathbf{x}||_1$ and $D\mathbf{x} \neq \mathbf{x}$ follows.

(iii): The assumption implies $||\mathbf{x}^+||_1 = ||\mathbf{x}^-||_1 = ||\mathbf{x}||_1/2$. Thus (i) implies $||D\mathbf{x}||_1 \le ||\mathbf{x}||_1 - \delta n||\mathbf{x}||_1 = (1-\delta n)||\mathbf{x}||_1$. □

Proof of Theorem 2. Fix k and δ as in Lemma 3.

The system of equations $\mathbf{x} = P\mathbf{x}$ is homogeneous and its rank is at most $n-1$: the column sums of P are equal to 1, so summing all the inequalities yields a trivial equality $\sum_{i=1}^n x_i = \sum_{i=1}^n x_i$. Thus there exists a non-trivial solution $\mathbf{x}$.

We claim that any such $\mathbf{x}$ has all entries non-negative or all the entries non-positive. Otherwise Lemma 5(ii) for $D = P^k$ implies $P^k\mathbf{x} \neq \mathbf{x}$, which is a contradiction with $P\mathbf{x} = \mathbf{x}$.

Since $\mathbf{x}$ is non-trivial and has no entries with opposite signs, a scaled vector $\boldsymbol{\pi} = \mathbf{x}/(\sum_{i=1}^n x_i)$ is a distribution. Since $\boldsymbol{\pi} = P\boldsymbol{\pi}$, $\boldsymbol{\pi}$ is a stationary distribution. Furthermore, it is unique: For any stationary distribution $\boldsymbol{\pi}' \neq \boldsymbol{\pi}$, the vector $\mathbf{y} = \boldsymbol{\pi} - \boldsymbol{\pi}'$ would be a non-trivial solution of the system of equations $\mathbf{y} = P\mathbf{y}$ with both positive and negative entries and we have already excluded the existence of such a solution.

Now consider an arbitrary initial distribution $\mathbf{p}$. We prove that $\lim_{t\to\infty} P^t\mathbf{p} = \pi$. Considering $\mathbf{p}$ equal to the jth standard basis vector, this implies that for each i, j, $P^t_{i,j}$ converges to π_i.

Consider an arbitrary distribution $\mathbf{q}$. For $s = 0, 1, 2, \cdots$, consider the vectors $\mathbf{v}^{(s)} = P^{sk}\mathbf{q} - \boldsymbol{\pi}$. We first prove that $\lim_{s\to\infty} \mathbf{v}^{(s)} = 0$. Since $\boldsymbol{\pi}$ is a stationary distribution, we have $\mathbf{v}^{(s)} = P^{sk}\mathbf{q} - \boldsymbol{\pi} = P^{sk}(\mathbf{q} - \boldsymbol{\pi})$ and $\mathbf{v}^{(s+1)} = P^k\mathbf{v}^{(s)}$. Note that the coordinates of each of $\mathbf{v}^{(s)}$ sum to 0. Using also the fact that $P^k_{i,j} > \delta$ by Lemma 3, we can apply Lemma 5(iii) to obtain

$$||\mathbf{v}^{(s+1)}||_1 = ||P^k\mathbf{v}^{(s)}||_1 \le (1-\delta n)||\mathbf{v}^{(s)}||_1 .$$

Thus

$$||\mathbf{v}^{(s)}||_1 \le (1-\delta n)^s||\mathbf{v}^{(0)}||_1$$

which converges to 0. Thus $\mathbf{v}^{(s)}$ converges to $\mathbf{0}$ and $P^{sk}\mathbf{q}$ converges to $\boldsymbol{\pi}$ for $s \to \infty$.

Now consider $\mathbf{q} = P^\ell\mathbf{p}$ for $\ell = 0, \ldots, k-1$. The previous paragraph implies that for each ℓ, the sequence $P^{sk}\mathbf{q} = P^{sk+\ell}\mathbf{p}$ converges to $\boldsymbol{\pi}$ for $s \to \infty$. This implies that also $P^t\mathbf{p}$ converges to $\boldsymbol{\pi}$ for $t \to \infty$. □