A Simple Analysis of the Harmonic Algorithm for Two Servers

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Abstract

HARMONIC is a randomized k-server algorithm that chooses the servers with probabilities proportional to the inverses of distances to the request point. We present a simple proof that HARMONIC is 3-competitive for $k = 2$.

1 Introduction

In the *k-server problem* we are given a metric space M populated by k mobile serves. At each time step a request $r \in M$ is specified, and we need to move one server to r to "satisfy" the request. The decision as to which server to move needs to be made *online*, without the knowledge about future requests.

An online server algorithm $\mathcal{A}$ for the k-server problem is called *C-competitive* if its cost on any request sequence ϱ is at most C times the optimal (offline) cost on this sequence, plus possibly an additive constant independent of ϱ. The same definition applies to randomized algorithms as well, we just need to replace our cost by our *expected* cost on ϱ. The *competitive ratio* of $\mathcal{A}$ is the smallest C for which $\mathcal{A}$ is C-competitive.

Manasse, McGeoch and Sleator [6] gave a 2-competitive algorithm for 2 servers and they proved that no deterministic online algorithm for k servers can be better than k-competitive. They also formulated the still-open *k-server conjecture* that a k-competitive algorithm exists for each k. The best known upper bound on the competitive ratio for $k \geq 3$ is $2k-1$, by Koutsoupias and Papadimitriou [5]. We refer the reader to [2, 4] for more general information on the k-server problem.

HARMONIC is a randomized k-server algorithm proposed by Raghavan and Snir [7]. If the distance between the server s_j and the request point is ℓ_j, the algorithm moves s_j to the request point with probability proportional to its inverted distance to the request, that is $\ell_j^{-1}/(\sum_i \ell_i^{-1})$. HARMONIC is natural, simple and memoryless. In spite of intense study, its exact competitive ratio for $k \geq 3$ is still unknown. The best upper bound, by Bartal and Grove [1], is exponential in k (see also [2]), much larger than the conjectured ratio and the known lower bound of $k(k+1)/2$ [7]. We note that the upper bound holds even against the so-called *adaptive adversary* that can choose requests based on the information on our server locations.

The case of $k = 2$ was settled by Chrobak and Larmore [3], who proved that for two servers HARMONIC is 3-competitive, thus matching the lower bound. The "lazy" potential function used in

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[3] was obtained by computing HARMONIC's maximal cost when the adversary alternates requests on his current server positions. The resulting potential is a complicated rational function, and the analysis involves tedious calculations.

In this note we present an alternative, simple proof that HARMONIC is 3-competitive for two servers. Our potential function is very simple: the distance between the unmatched servers, plus twice the distance from our unmatched server to the adversary's closest server. This function can be thought of as an approximation to the potential from [3]. It converges to the lazy potential when the servers are almost matched, and extends in a piecewise linear fashion to all other configurations. We believe that, by studying such approximations for larger values of k, it should possible to derive simple potential functions for proving the optimal ratio (or, at least, a more accurate upper bound).

2 Analysis of HARMONIC

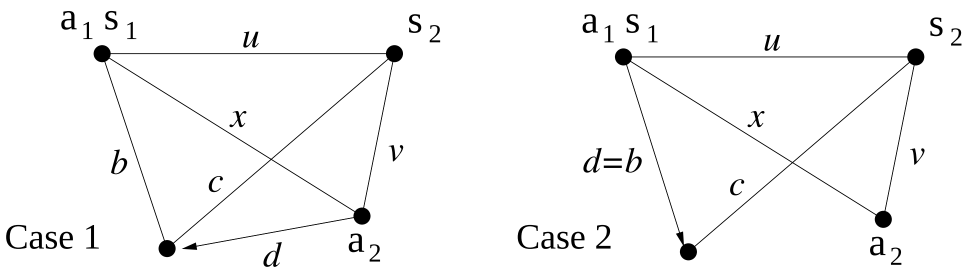
Denote our servers by s_1, s_2 and the adversary servers by a_1, a_2. We divide the game into steps. In one step the adversary moves one server to a new point and requests it. Then our algorithm serves the request. Without loss of generality, assume that the last request was served by a_1 and s_1; thus they are at the same point. Let u be the distance between a_1 and s_2 and v the distance between a_2 and s_2. We define the potential of this configuration as

$$\Phi_{uv} \quad = \quad v + 2\min\{u, v\}.$$

Suppose that in the next step the adversary moves his server by distance d. We want to prove that the potential function satisfies

$$cost_H + \Phi' \quad \leq \quad 3d + \Phi_{uv} \tag{1}$$

where $cost_H$ is the expected cost of HARMONIC in this step and Φ' is the expected value of the new potential. Let x denote the distance between a_1 and a_2 (before the move) and let b and c denote the distance of the request point to s_1 and s_2, respectively. We have $cost_H = 2bc/(b+c)$. We distinguish two cases, depending on which adversary server moved to the request point (see the figure below).



Case 1: The adversary moves a_2. Since, by the triangle inequality, $\Phi_{uc} \leq 3d + \Phi_{uv}$, it is sufficient to prove that $cost_H + \Phi' \leq \Phi_{uc}$. The expected value of the new potential is $c\Phi_{cu}/(b+c)$, and after substituting and rearranging the last inequality is equivalent to

$$c(b + u - c) \quad \leq \quad 2b\min\{u, c\} \tag{2}$$

Using triangle inequalities involving b, c and u, we have $c(b+u-c) \leq c(b+b) = 2bc$ and $c(b+u-c) = (c-u)(b-c) + bu \leq 2bu$, so (2) holds.

Case 2: The adversary moves a_1. Thus $d = b$. The expected potential value is $c\Phi_{cv}/(b+c)+b\Phi_{bx}/(b+c)$. We have $\Phi_{bx} \leq x + 2b \leq u + v + 2b$. Using this and rearranging, inequality (1) reduces to

$$bu + 2c \min\{v, c\} \quad \leq \quad b^2 + bc + 2(b + c) \min\{u, v\} \tag{3}$$

Since $u \leq b+c$, the left side is at most $b(b+c)+2cv \leq b^2+bc+2(b+c)v$. On the other hand, using triangle inequalities, we can also bound the left side by $bu + 2c^2 \leq bu + 2(b + u)c \leq bu + bc + b(b + u) + 2uc = b^2 + bc + 2(b + c)u$. Thus (3) holds.

We proved that (1) holds in each step. Thus, by amortizing over the whole request sequence, we obtain the following result.

Theorem 1 HARMONIC *is 3-competitive for 2 servers against the adaptive adversary.*

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