The optimal absolute ratio for online bin packing*

János Balogh[†] József Békési[†] György Dósa[‡] Jiří Sgall[§] Rob van Stee[¶]

Abstract

We present an online bin packing algorithm with absolute competitive ratio 5/3, which is optimal.

1 Introduction

In the online bin packing problem, a sequence of *items* with sizes in the interval $(0, 1]$ arrive one by one and need to be packed into *bins*, so that each bin contains items of total size at most 1. Each item must be irrevocably assigned to a bin before the next item becomes available. The algorithm has no knowledge about future items. There is an unlimited supply of bins available, and the goal is to minimize the total number of used bins (bins that receive at least one item).

Bin packing is a classical and well-studied problem in combinatorial optimization. The *offline* version, where all the items are given in advance, is well-known to be NP-hard [6]. Extensive research has gone into developing approximation algorithms for this problem. Such algorithms have provably good performance for any possible input and work in polynomial time. In fact, the bin packing problem was one of the first for which approximation algorithms were designed. The (absolute) approximation ratio of an algorithm is the worst case ratio, over all possible inputs, of its cost for a particular input divided by the optimal cost for the same input. Simchi-Levi [11] showed that First Fit Decreasing and Best Fit Decreasing have the best possible absolute approximation ratio of 3/2. For surveys, see [2, 3].

The focus of the research into approximation algorithms is on the question of how much performance degrades if an algorithm is constrained to work in polynomial time. In practical packing problems, however, it happens frequently that the input is not known completely before the algorithm starts working. It is therefore very natural to consider the *online* version of this problem. In online problems, we ask how much performance degrades as a result of not knowing the future. In general, there is no restriction on the amount of computation time used by an online algorithm. However, most online algorithms, including all the ones we consider in this paper, are very efficient.

For an input L, let $ALG(L)$ be the number of bins used by algorithm ALG to pack this input. Let $\text{OPT}(L)$ denote the number of bins in an optimal solution. The most common performance measure for online bin packing algorithms is the asymptotic performance ratio, or asymptotic competitive ratio, which is defined as

$$(1.1) \quad R_{ASY}(A) := \limsup_{n\to\infty} \left\{ \max_{L:\text{OPT}(L)=n} \left\{ \frac{A(L)}{n} \right\} \right\}.$$

Hence, for any input L, the number of bins used by an online algorithm A is compared to the optimal number of bins needed to pack the same input. Note that calculating the optimal number of bins might take exponential time; moreover, it requires that the entire input is known in advance.

One of the most famous algorithms for bin packing is an online algorithm called First Fit (FF). It packs each item into the first bin where it fits. First, Ullmann [12] proved that the asymptotic competitive ratio of FF is 1.7. Later, Garey et al. [7] and Johnson et al. [8] extended this work. Among other results, they proved that FF works much better if the elements of the input are sorted in decreasing order. In this case the asymptotic performance ratio is $\frac{11}{9}$. Of course, this algorithm cannot be used for the online problem. Later, algorithms improving on FF were given, for example Harmonic Fit by Lee and Lee [9]. The current best online

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[†]Department of Applied Informatics, Gyula Juhász Faculty of Education, University of Szeged, H-6701 Szeged, POB 396, Hungary. `{balogh,bekesi}@jgypk.u-szeged.hu`

[‡]Department of Mathematics, University of Pannonia, H-8200 Veszprém, Hungary. `dosagy@almos.vein.hu`

[§]Computer Science Institute of Charles University, Faculty of Mathematics and Physics, Praha, Czech Republic. `sgall@iuuk.mff.cuni.cz`

[¶]Department of Computer Science, University of Leicester, Leicester, UK. `rob.vanstee@leicester.ac.uk`

algorithm was given by Seiden in 2002 [10], it is based on the idea of Harmonic Fit and it has an asymptotic competitive ratio of 1.58889.

Van Vliet [13] proved that there is no online algorithm with asymptotic competitive ratio below 1.54014. Recently this was improved to $\frac{248}{161} = 1.54037$ by Balogh et al. [1].

Definition (1.1) focuses on the long-term behavior of online algorithms. For small inputs, the relative performance of an online algorithm might be worse than the asymptotic performance ratio suggests. If we want to have a performance guarantee relative to the optimal solution for every possible input, we need to consider the absolute competitive ratio, which is defined as follows:

$$R_{ABS}(A) := \sup_L \left\{ \frac{A(L)}{\text{OPT}(L)} \right\}. \tag{1.2}$$

The absolute competitive ratio of FF was only recently determined to be 1.7 (equal to the asymptotic ratio) by Dósa and Sgall [4]. Afterwards, the absolute competitive ratio of Best Fit (BF), which packs each item into the bin where it leaves the least amount of space unused, was also shown to be exactly 1.7 [5]. Before our work, 1.7 was the best known absolute competitive ratio of any algorithm. There is a simple lower bound, consisting of only 18 items, which shows that no algorithm can be better than 5/3-competitive. This folklore result is included in Section 2 for completeness.

The natural question is then whether an algorithm with better ratio than FF or BF exists. In this paper, we answer this question in the affirmative by presenting an online algorithm with absolute competitive ratio 5/3.

To give some intuition, consider the worst case for FF which is given by instances of the following form. The input starts with $10k$ items of sizes very close to 1/6 (for some integer k). FF packs these items into $2k$ bins. Then, $10k$ items of sizes very close to 1/3 arrive. These items are packed in pairs into $5k$ bins by FF. Finally, $10k$ items of size slightly more than 1/2 arrive, that FF packs into individual bins. In the end, $FF = 17k$, while the items can be packed into $10k$ bins.

In these instances, it is notable that all bins used by FF are relatively full, apart from the last $10k$ bins. In our algorithm, we try to avoid the bad situation where, at the end of the input, the algorithm has to open many new bins that are only half full.

Definitions. The *level* of a bin is the sum of the sizes of the items in it. A bin is called a *k-bin* if there are exactly k items packed into it. A *k^+-bin* contains *at least* k items. Items of size more than 1/2 are called *large*, others are called *small*. At any time during the execution of the algorithm, a bin is called *open* if it can be used to pack later items. Let T denote the total size of the items.

Idea of the Algorithm. The main idea of our algorithm, which we call Five-Thirds (FT), is to use FF whenever possible, but try to avoid long sequences of 2-bins. It can be shown easily that 3^+-bins are generally fuller than 2-bins, and this compensates for 1-bins that are only half full at the end. However, 2-bins are problematic, since they may be only about 2/3 full on average. Therefore, whenever FT is about to put an item into any bin that has one item so far, it will from time to time put such an item into a new, empty bin instead, creating a *special bin* which is *specifically reserved for a large item*; no other item will be packed into it.

Since it is possible that no large items arrive after all, FT needs to be conservative about creating these special bins, since they are initially less than half full. In fact, we need to be extremely careful about the conditions for creating new special bins, in order to be able to deal with any possible input.

Overview of the Analysis. In Section 3 we present the algorithm FT and some basic properties of the produced solutions. The analysis of FT then splits into three main cases. A major difficulty in all cases is that there may exist a single non-special 1-bin that has an item of size less than 1/2 (e.g., if this item arrives near the end of the input and is followed only by large items that do not fit with it). This complicates both the size-based and the weight-based analysis methods that we describe below.

If no special bin is ever created, FT behaves as FF throughout, and (due to our conditions for creating special bins), FF is 5/3-competitive in this situation (Section 4). This case is relatively easy, but note that it includes the instances that prove the tight lower bound. The main technical difficulties occur in the next two cases.

If there exists a special bin that does not contain a large item at the end of the input (Section 5), it means that all large items in 1-bins are relatively large, since FT always puts large items in existing special bins and special items in existing 1-bins with large items if they fit. For this case, we use a size-based analysis: essentially, this boils down to showing that the bins of FT are on average at least 3/5 full. The way of proving this depends on how many 1-bins there are compared to the number of special bin. At the end, some small cases need to be examined in great detail to get the desired result.

Finally, if all special bins have large items (Section 6), it means that these bins are relatively full. For this case, we use a weight-based analysis. Each item is assigned a weight which is a measure for how much space this item needs in any packing. In order to prove an upper bound of 5/3 instead of 1.7, we had to modify the weight function used by Dósa and Sgall [4]. The idea is that each optimal bin has total weight packed into it of at most 5/3, whereas FT packs an average weight of at least 1 per bin, immediately implying the desired result. Proving this requires a significant amount of work and hinges on the precise conditions for creating special bins. In the end, the weight-based analysis leaves one case open, where only one special bin is created; for this case, we use size-based arguments again.

2 Lower bound of $5/3$

First, 6 items of size 1/7 arrive. If they are packed in more than one bin, the competitive ratio is at least 2 and the input stops.

Otherwise the 6 items are packed in a single bin. Then 6 items of size $1/3+\varepsilon$ arrive for some $\varepsilon \in (0, 1/42)$. They do not fit in the first bin. If they are packed into four or more bins, the competitive ratio is at least 5/3 and the input stops.

Otherwise the only possibility is to pack the 6 items of size $1/3+\varepsilon$ into three bins, two per bin. Then, 6 items of size $1/2+\varepsilon$ arrive. These items must be packed into 6 separate new bins, for a total of 10 bins. However, the entire input can be packed into only 6 bins with one item of each size, proving the lower bound.

3 The algorithm

At each time, FT maintains a partition of open bins into a set of **special** bins and a set of **regular** bins. A regular bin can become a special bin, but any special bin stays special until the end of the instance (it may become a completed special bin). Furthermore, the following invariant holds.

Invariant 3.1. *A* special bin *holds exactly one item smaller than* $1/2$ *(which is called a* special item*) and at most one other item, which must be large. A special bin with one item (the special item) is* open *and a special bin with two items is* completed.

Thus, special bins that have only one item are specifically reserved for large items that may arrive later, and will not receive any other item. A special bin can be created in three ways, all of which we later use in our algorithm: (i) a regular bin with a single small item can be declared special, (ii) upon creating a new bin with a small item, this bin can be declared special, and (iii) when packing a small item in a bin with a single large item, this bin can be declared special.

Regular bins are always open and can receive any item (which fits, of course).

Let s denote the number of special bins.

Definition 3.2. *A regular* 2^+*-bin is called* good *if at least one of the following conditions holds:*

- *it has level at least 5/6,*
- *it contains a large item, or*
- *the total size of its first two items is at least 3/4.*

A regular 2*-bin is called* critical *if it is not a good bin. A regular* 2+*-bin is called* interesting *if it does not contain a large item.*

All types of good bins will be straightforward to analyze, as we will see. It is easy to see that a good bin stays good and regular if more items are packed into it. On the other hand, only 2-bins can be critical. Hence, a bin can become critical (only) when the second item is packed into it, but a critical bin loses this property if the bin receives a third item later.

Ordering. Our algorithms will open bins one by one. Whenever we speak about first, last, next bin (etc.), or when we consider an ordered list of bins, we will always be referring to the order in which bins were opened by our algorithm.

Matching. In our analysis, we will often consider pairs of bins, where each pair will have a total packed size strictly more than 1. We will say that the bins of such a pair are *matched* to each other.

One type of matching is done by FT itself. This matching additionally guarantees that the matched bins have large total weight (see Section 5 for a definition of the weights). Whenever this algorithm creates a special bin, it will immediately be *matched* to an existing critical bin, which is not yet matched to any special bin. To be precise, we always use the *last* such critical bin. In the algorithm, as a necessary condition for creating a special bin we require that such a critical bin exists, so the matching is well-defined.

As we mentioned above, a critical bin can stop being critical, but if it was matched to some special bin it will remain matched.

A detailed description of the algorithm FT is shown in Figure 1. In Step 2, we decide whether we should create a new special bin or not. First we check what FF would do with the current item, while "hiding" the special bins from FF. That is, FF makes its decision based only on the current sequence of regular bins and the item a.

For each item a of the input, do the following:

1. If a is large, pack a by FF into the collection of all bins (or into a new bin if it does not fit anywhere).
2. Otherwise, let B be the bin that FF would pack a into if the collection of existing bins were restricted to the regular bins. (B is possibly a new bin.) If after packing a into B, there are at most $\max(3, 4s + 1)$ *interesting* bins, or B is not critical, or no unmatched critical bin exists, pack the item into B.
3. Otherwise, if before packing a there exists a regular 1-bin with a large item where a fits, pack a into the first such bin B'; declare B' to be a special bin where a is the special item. Match the special bin to the last unmatched critical bin.
4. Otherwise pack a into a new bin B''. Let b be the only item previously packed in B. If $a \leq b$, let B'' be a special bin while B remains regular. Else, change B to a special bin and let B'' be regular. The single item packed into the special bin is a special item. Match the special bin to the last unmatched critical bin.

Figure 1: Algorithm FT

The bound $\max(3, 4s + 1)$ means that after a slightly irregular initial phase, roughly every *fifth* time that FT opens a new bin, one bin (the new one, or an existing one) is turned into a special bin instead (but only if all the conditions are satisfied, of course). See below for an example.

If the bin B used by FF is acceptable (that is, one or more of the conditions listed in Step 2 hold), then we pack the new item into B in this Step. If a new special bin is created, this is done in Step 3 or in Step 4.

In Step 3, note that $B' \neq B$, because B' is not critical and B would be critical if a were packed into it. In fact, B' comes after B, because a fits in B' but FF did not suggest it. Regarding Step 4, note that item b is indeed unique: if a would have been packed into B, we know by Step 2 that B would have been critical; in particular, B would have been a 2-bin. Hence, B must have been a 1-bin before.

An item always gets packed in Step 2 (by using FF) if the bin B proposed by FF does not become critical after packing a, e.g., if a opens a new bin. Moreover we do not make any special bin as long as there are at most three regular bins. The earliest possible time that a special bin could be created is when FF would create the fourth regular 2^+-bin, it is a critical 2-bin, and another critical bin already exists (among the three existing regular bins). After creating the first special bin, we allow at least $4s + 1 = 5$ regular bins in total before possibly creating another special bin.

It is possible that it takes much longer for the first special bin to be created, and there are not always four regular bins between two successive special bins (or two regular bins between the first and second special bin).

As an example, it could happen that first, a large number of non-critical regular (but interesting) bins are opened. Then, a bin B_1 becomes critical by receiving its second item a_1. At this point, no unmatched critical bin is available, so still no special bin is created. For the next item a_2, a new bin B_2 is opened, and it is turned into the first special bin when item a_3 arrives (which is packed into its own new bin B_3). The bin B_2 is matched with B_1. Item a_4 is then packed into B_3, making it critical, but not special, because again no unmatched critical bin exists. Then, another bin B_4 is opened for item a_5, and it is turned into the second special bin (matched with B_3) when the next item a_6 arrives (which is packed into its own new bin B_5). In the end, the special bins B_2 and B_4 are separated by only one regular bin in this example.

Lemma 3.3. *All items in regular bins are packed by the FF rule. More precisely, if we remove all the items packed into special bins from the instance and run FF on this instance, we get the same packing as in the regular bins of the original instance.*

Proof. It is sufficient to show that any item a packed into a regular bin A does not fit into any previous regular bin. Inductively, we can assume that all previous items are packed by FF. If a is packed in Steps 1 or 2, it is packed by FF into a regular bin and the claim follows. In Step 3, no item is packed into a regular bin.

It remains to handle the case when a is packed in Step 4 and a is packed into a new regular bin. In this case, FF packs a into B which contains a single item $b < a$, and FT makes B a special bin. Since FF packs a into B, it follows that a does not fit into any bin before B. Now consider any regular bin C after B and an item c in C: Since c does not fit into B, we have $1 < b + c < a + c$ and thus in turn a does not fit into C, so a indeed does not fit into any existing regular bin. ■

LEMMA 3.4. *At any time when FT creates a special bin, each regular bin except for one either has a large item, or it is interesting and will remain so until the end of the input.*

Proof. Consider a time when FT creates a special bin, say when item a arrives. This means that FF suggested to pack item a into a regular 1-bin B with a small item b, making it critical. Bin B can only have been opened at a time when b fit in no earlier *regular* bin by Lemma 3.3, so all preceding regular bins are more than half full. In particular, all preceding regular 1-bins contain a large item. All regular bins that were opened after B was opened and while B was still a 1-bin (if any) contain a large item, and are therefore not interesting. Therefore, when item a arrives, all of the existing interesting bins precede B, are more than half full and cannot receive a large item later, so they remain interesting until the end of the input. Bin B itself and the new bin opened for a are not interesting at this time, but one of them is made special. ■

For the analysis, we assume that there exists an input L for which $FT(L) > \frac{5}{3}\textsc{opt}(L)$. We fix L and abbreviate $FT(L)$ by FT and $\textsc{opt}(L)$ by $\textsc{opt}$. Because FT and $\textsc{opt}$ are integers, our assumption is equivalent to assuming

$$FT \geq \frac{5}{3}\textsc{opt} + \frac{1}{3}\,. \tag{3.3}$$

In all cases, we will derive a contradiction. We divide the regular bins into several groups according to their final contents as follows. A regular bin is called a *dedicated* bin, if the bin contains only one item (after having run the execution of the algorithm). Such an item is called *singular*. Let the number of dedicated bins (for the fixed input L) be denoted by δ. If there exists a dedicated bin with level at most $1/2$ (we will see later that there can be at most one such bin) the (small) item in this bin is denoted by d_0, otherwise d_0 is undefined. The number of small and large singular items are denoted by δ_0 and δ_1, respectively. Then naturally,

$$\delta = \delta_0 + \delta_1.$$

Lemma 3.3 immediately implies that there can be at most one small singular item, i.e. $\delta_0 \leq 1$. If there were two small singular items (in two regular bins), then FT would pack them together instead, or one of them would be defined to a special item. It also follows that $\delta \leq \textsc{opt}$, as no bin in an optimal solution can contain two singular items.

LEMMA 3.5. *At any time during the procedure, if x is the size of some large item that is packed alone in a bin, and y is the size of a special item in some open special bin, then $x + y > 1$.*

Proof. If FF suggests to pack a new item with an existing large item, or a large item into any existing bin, this proposal is always accepted. If the proposal of FF is not accepted, we first consider existing bins with large items in Step 3 before packing the new item in a new bin. ■

LEMMA 3.6. *Let J be a set of regular bins of size $j = |J| \geq k + 1$, of which at least the last $j - k$ bins are regular k^+-bins. Then the total level of the bins in J is more than*

$$\frac{jk}{k+1}\,.$$

Proof. Consider any $k + 1$ bins of J with the smallest levels. It is sufficient to see that their total level is more than k, as the lemma then follows by averaging over all $(k + 1)$-tuples of bins of J. Let the $k + 1$ bins in consideration be denoted by $B_1, \ldots, B_{k+1}$ in the order of the packing of FT. Note that B_{k+1} is a k^+-bin by the assumption of the lemma. Let the smallest level among $B_1, \ldots, B_k$ be x. Then any item in B_{k+1} is larger than $1 - x$, thus the total level of the $k + 1$ bins is more than $kx + k(1 - x) = k$. ■

LEMMA 3.7. *The total size of a special item and the items in the bin it is matched to is more than 1. Each special item has size more than 1/4 but less than 3/8.*

Proof. Consider the step when a special bin is created, i.e., when packing item a, FF suggests a 1-bin B with item b which is not large (since packing a into B would make B a critical bin). At this time, there can be no critical bins following B since the items in such bins, not being large, would have fit in B (cf. the proof of Lemma 3.4). So the critical bin C that is matched to the new special bin must be before B. But FF did not suggest it for a, so a does not fit there. This proves that the total size of a and the items in C is more than 1 and also that $a > 1/4$, as C is critical at this point and thus has two items of total size less than $3/4$. Furthermore, also b does not fit into C, as B and C are both regular bins, thus also $b > 1/4$.

Since one of a and b becomes the special item, we have proved the lower bound on the special item. For the upper bound, we know that $a + b < 3/4$ if a special bin is created. If it is created in Step 4, the smaller one of a and b becomes the special item, and its size is at most $3/8$. If a is packed into a dedicated bin D in Step 3 and becomes a special item, note that b does not fit in D, as both B and D are regular 1-bins before this step and were packed by FF (Lemma 3.3). Thus $a < b$ and $a < 3/8$ follows as well. ■

4 No special bin is created

THEOREM 4.1. *If no special bin is created then*

$$FT \le \frac{5}{3}\text{OPT}.$$

Proof. Suppose that no special bin is created. It follows that at any step of algorithm FT we accepted the proposal of FF, thus finally we get just the FF packing. For FF, we already know that $FF \le 1.7 \cdot \text{OPT}$ holds.

If $\text{OPT} \le 9$, the stronger inequality $FF \le \frac{5}{3}\text{OPT}$ is true according to Table 1.

Now suppose that $\text{OPT} \ge 10$. For this case, we have the following lemma.

LEMMA 4.2. *There can be only at most one 2^+-bin B_0 that is not a critical 2-bin and has level below 3/4. Moreover if such a bin exists, it has an item smaller than 1/4.*

Proof. Suppose there exists such a bin. If B_0 is a 2-bin, then it has a large item, since the bin is not critical. Then the bin also has an item $a < 1/4$, since the level of the bin is below 3/4. If B_0 is a 3^+-bin, it contains at least three items, and there is again an item $a < 1/4$ in the bin. In both cases it follows the level of any earlier bin is larger than 3/4, thus there cannot be two such bins. ∎

Let c be the number of critical bins. Let $q_1 = 1$ if B_0 exists and has no large item, else $q_1 = 0$. Note that the level of any further 2^+-bin is at least 3/4, by Lemma 4.2. Let Q_2 be set of the remaining 2^+-bins, and $q_2 = |Q_2|$ their number. Recall that δ is the number of 1-bins (including the bin of d_0 if it exists). For the remainder of this section only, if B_0 contains a large item, we also count B_0 as a 1-bin, i.e., we increase δ by 1.

Then $FF = \delta + q_1 + q_2 + c$. Assume (3.3). If $\text{OPT} \ge 10$, we get for the number of 2^+-bins

$$q_1 + q_2 + c = FF - \delta \ge \frac{2}{3} \cdot 10 + \frac{1}{3} = 7, \tag{4.4}$$

using $\delta \le \text{OPT}$.

LEMMA 4.3. *We have $c + q_1 \le 3$.*

Proof. If B_0 exists, there is no critical 2-bin before it, because B_0 contains an item of size at most 1/4 by Lemma 4.2. If B_0 has no large item, there also cannot be three critical bins following B_0, because the last one would be at least the fourth interesting bin overall, and FT would have created a special bin. It follows that if $q_1 = 1$, then $c \le 2$. If $q_1 = 0$, then $c \le 3$ because FT never creates a special bin. ∎

Our plan is to apply Lemma 3.6 on the critical bins and B_0 if B_0 contains no large item. As noted above, we already have a level guarantee of 3/4 for the bins of Q_2. In order to apply Lemma 3.6, we need at least three bins. We can ensure this by the next procedure.

Ensuring $c = 3$ and $q_1 = 0$. If $c = 3$ already holds, then $q_1 = 0$ by Lemma 4.3, and we are done. If $c < 3$, we are going to increase this sum (i.e. $c + q_1$) to 3 by counting bin B_0 (which is a 2^+-bin), and if needed, some other 2^+-bins from Q_2 as critical bins and decreasing the corresponding counter q_2. Note that by (4.4), $q_1 + q_2 \ge 7 - c > 3$ in this case. Hence, we can always find sufficiently many bins to ensure that $c = 3$. Moreover, $q_1 = 0$ holds at the end of the procedure.

Ensuring a level guarantee of $\delta/2$ for the 1-bins. There is nothing to do if $\delta = 0$ or $\delta \ge 2$. If $\delta = 1$, it could happen that the single dedicated bin does not contain a large item and has level below 1/2. However, the total level of the dedicated bin and any other bin is more than 1. Let us choose a bin from the set Q_2 (which was still not empty before this step, see above), making a match for the dedicated bin, so their total level is more than 1, and let these two bins be counted into δ, decreasing appropriately the number of the bins taking into account in q_2. Now $\delta = 2$.

We can now apply Lemma 3.6 (which gives a sharp bound) on the $c = 3$ 2^+-bins that we selected, and use the level guarantee of 3/4 for the 2^+-bins in Q_2. Recall that the total size of all the items is denoted by T. It follows that

$$\begin{aligned}\text{OPT} \ge T &> \frac{3}{4}(FF - \delta - c) + \frac{2}{3}c + \frac{1}{2}\delta \\ &\ge \frac{5}{4}\text{OPT} + \frac{1}{4} - \frac{1}{4}\delta - \frac{1}{12}c \ge \text{OPT},\end{aligned}$$

using $c = 3$ and $\delta \le \text{OPT}$, a contradiction. ∎

5 There exists an open special bin

The analysis in this section is based on the following lemma.

LEMMA 5.1. *If there exists an open special bin after all items have been packed, all large items in dedicated bins have size more than 5/8.*

Proof. This follows from Lemma 3.5 and Lemma 3.7. ∎

We will make use of this fact and match some special items to large items instead of using the matching of FT. Assume (3.3). Let $FT = \frac{5}{3}\text{OPT} + 1/3 + x/3$ with some integer $x \ge 0$. Recall that $\delta = \delta_1 + \delta_0$, where $\delta_0 = 1$ if d_0 exists and it is 0 otherwise. The following Observation follows immediately from the definition of FT and Lemma 3.4.

OPT	1	2	3	4	5	6	7	8	9
$1.7 \cdot$ OPT	1.7	3.4	5.1	6.8	8.5	10.2	11.9	13.6	15.3
$\lfloor 1.7 \cdot \text{OPT} \rfloor$	1	3	5	6	8	10	11	13	15

Table 1: Calculation of the competitive ratio of FF for small instances.

OBSERVATION 5.2. *At any moment of the procedure, if there are $s > 1$ special bins, then there are at least $4(s-1)+1$ interesting bins. If $s = 1$, then we have at least 3 such bins.*

Now let q denote the total number of regular 2^+-bins (including the bins with large items) in consideration. Then

$$q = 4(s-1) + 1 + k = 4s + k - 3,$$

where $0 \le k$ and $FT = q + s + \delta$. (For $s = 1$, we have $k \ge 2$.) Writing FT in two forms as

$$\begin{aligned} FT &= \frac{5}{3}\text{OPT} + 1/3 + x/3 \\ &= (4s - 3 + k) + s + \delta, \end{aligned}$$

we get

$$5\text{OPT} = 15s - 10 + 3k + 3\delta - x. \tag{5.5}$$

This equation in particular implies

$$3k + 3\delta \equiv x \pmod 5. \tag{5.6}$$

We show in the following that there is no solution to these equations. To do this, we define some additional matchings between bins.

Case a: There are $0 \le \delta_1 \le s$ large singular items. Consider the moment when the first special bin is created. At this time there existed at least three interesting bins, from which at least one was critical, and was used as the match of the first special bin. Let us choose another bin among these three bins, and denote it as B. We match the dedicated bin of d_0 (if $\delta_0 = 1$) to bin B. Next, we find a match for any large singular item among the special bins. As the special bins already do have matches, we first remove the matches of any δ_1 special bins, and then we match any such special bin to a large singular item.

There remain $s - \delta_1 \ge 0$ special bins, that we did not use for making matches for the large singular items. Among them, there are say $s_1 \ge 0$ completed special bins, and s_2 open special bins. Then $s - \delta_1 = s_1 + s_2$. We also remove the matches of the completed special bins, and keep only the ones for the open special bins.

LEMMA 5.3. *We have $s_2 + \delta_1 + \delta_0$ pairs of matched bins. Each pair has total level more than 1.*

Proof. The number of pairs follows from the description above. The bound on the level holds if the special bin contains a large item (the pair has two large items in this case), and for the pair with d_0 since FF did not pack the items in this pair in one bin. For each open special bin that is matched with a large item, by Lemma 3.5, the large item does not fit together with the special item. For the other open special bins, we apply Lemma 3.7. ■

There remain $q - s_2 - \delta_0$ bins among the 2^+-bins that are not used for making matches. If $q - s_2 - \delta_0 \ge 3$, we can apply Lemma 3.6 for these bins. We note that the total level of the completed special bins is more than $3s_1/4$. Using Lemma 5.3, we get

$$\begin{aligned} 3\text{OPT} &\ge 3T \\ &> 3\left(\frac{2}{3}(q - s_2 - \delta_0) + \frac{3}{4}s_1 + s_2 + \delta_1 + \delta_0\right) \\ &= 2q + \frac{9}{4}s_1 + s_2 + \delta_0 + 3\delta_1 \\ &\ge 2q + s_1 + s_2 + 3\delta_1 + \delta_0 \\ &= 2(4s + k - 3) + (s - \delta_1) + 3\delta_1 + \delta_0 \\ &= 9s + 2k + 2\delta_1 + \delta_0 - 6 \end{aligned}$$

which means (applying the integrality of OPT) that

$$15\text{OPT} \ge 45s + 10k + 10\delta_1 + 5\delta_0 - 25$$

which together with the first equality (5.5) gives

$$\begin{aligned} 45s - 30 + 9k + 9\delta - 3x &\ge 45s + 10k + 10\delta_1 + 5\delta_0 - 25 \\ &\Rightarrow 4\delta_0 \ge k + 3x + \delta_1 + 5, \end{aligned}$$

where $\delta_0 \le 1$. Since the right hand side is at least 5, we got contradiction.

Now assume that $q - s_2 - \delta_0 \le 2$. Since $q \ge 4s - 3$, $s_2 \le s$ and $\delta_0 \le 1$, we have $s \le 2$.

Case a1: $s = 1$. Since $q \le 2 + s_2 + \delta_0$, this means that

$$\begin{aligned} FT = q + s + \delta &\le (2 + s_2 + \delta_0) + 1 + \delta_1 + \delta_0 \\ &= (s_2 + \delta_1) + 2\delta_0 + 3 \le 6, \end{aligned}$$

since $s = \delta_1 + s_1 + s_2 = 1$ and $\delta_0 \le 1$. The total level of the three 2^+-bins just when the (only) special bin is created is more than 2 by Lemma 3.6, thus OPT ≥ 3. Then OPT $= 3$ and $FT = 6$, otherwise $FT/\text{OPT} \le 5/3$

holds, and we get a contradiction. This implies $\delta_0 = 1$ and $s_2 + \delta_1 = 1$.

If $\delta_1 = 1$, then the total level of the two dedicated bins is more than 1, the total level of three 2^+-bins is more than 2, thus $\text{OPT} \geq S > 2+1 = 3$, a contradiction. Thus $\delta_1 = 0$. It means that $q = 4$. Then the total level of the dedicated bin and one 2^+-bin is more than 1, the total level of the three other 2^+-bins is more than 2, thus $\text{OPT} \geq S > 2 + 1 = 3$, a contradiction.

Case a2: $s = 2$. Since $q - s_2 - \delta_0 \leq 2$, $q \geq 4s - 3$ and $s - \delta_1 = s_1 + s_2$, we must have $s_2 = s = 2$, $q = 5$, $\delta_1 = 0$ and $\delta_0 = 1$. We have $FT = q + s + \delta_1 + \delta_0 = 8$. Note that the second special bin was created at a time when q is already 5, its final value (it could not have happened earlier). This means that it must be the item d_0 that would make the special bin critical if they were packed together. So all three singular items have size more than 1/4. But then $\text{OPT} > \frac{2}{3}q + \frac{3}{4} > 4$, so $\text{OPT} \geq 5$, a contradiction.

Case b: $\delta_1 \geq s + 1$. First we find a match for the dedicated bin of d_0 if $\delta_0 = 1$, from large singular items. This is possible since $\delta_1 \geq s + 1 \geq 2$, and there remain at least s large singular items.

Let now $s = s_1 + s_2$, where there are $s_1 \geq 0$ completed special bins, and there are s_2 singular special items. By the assumption of Section 4, $s_2 \geq 1$.

Now we remove the matches of any singular special items, and then we find a match for any singular special item among the large singular items. The total level of any of these matches is strictly more than 1, and there are enough large singular items for making these s_2 different matches. There remain $\delta_1 - s_2 - \delta_0$ large singular items, any of them is strictly larger than 5/8 by Lemma 5.1.

Now we remove the matches also of any completed special bin, as we want to count the total size of these bins alone (without their matches), as the level of any such bin in itself is at least 3/4.

There are $q \geq 3$ such 2^+-bins that are not used for making matches. We apply Lemma 3.6, and we get

$$\begin{aligned}
&24\text{OPT} \geq 24T \\
(5.7)\qquad &> 24(\frac{2}{3}q + \frac{3}{4}s_1 + s_2 + \frac{5}{8}(\delta_1 - s_2 - \delta_0) + \delta_0) \\
&= 24(\frac{2}{3}q + \frac{3}{4}s_1 + \frac{3}{8}s_2 + \frac{5}{8}\delta_1 + \frac{3}{8}\delta_0) \\
&= 16(4s_1 + 4s_2 - 3 + k) \\
&\quad + 18s_1 + 9s_2 + 15\delta_1 + 9\delta_0 \\
&= 82s_1 + 73s_2 + 16k + 15\delta_1 + 9\delta_0 - 48
\end{aligned}$$

which means by the integrality of OPT that

$$24\text{OPT} \geq 82s_1 + 73s_2 + 16k + 15\delta_1 + 9\delta_0 - 47.$$

Together with (5.5), this gives

$$\begin{aligned}
120\text{OPT} &= 24(15s - 10 + 3k + 3\delta - x) \\
&= 360s_1 + 360s_2 + 72k + 72\delta_1 + 72\delta_0 - 240 - 24x \\
&\geq 5(82s_1 + 73s_2 + 16k + 15\delta_1 + 9\delta_0 - 47) \\
&= 410s_1 + 365s_2 + 80k + 75\delta_1 + 45\delta_0 - 235 \\
&\Rightarrow 27\delta_0 \geq 50s_1 + 5s_2 + 5 + 8k + 3\delta_1 + 24x.
\end{aligned}$$

This means that in a counterexample we have $\delta_0 = 1$, $s_1 = x = 0$. After this simplification, the inequality looks as follows:

$$(5.8)\qquad 22 \geq 5s_2 + 8k + 3\delta_1,$$

where we recall that $\delta_1 \geq s+1 \geq s_2+1$. By (5.6), $k+\delta$ is divisible by five. Now $\delta \geq \delta_0 \geq 1$, thus $k + \delta > 0$. By (5.8), it cannot be 10 or more. Thus $k + \delta = 5$, i.e. $k + \delta_1 = 4$. Then $8k + 3\delta_1 \geq 12$, and thus $s_2 \leq 2$.

Suppose $s_2 = 1$. (Since $s_1 = 0$, and $s \geq 1$, s_2 cannot be zero.) Then $\delta_1 \geq s + 1 \geq 2$. As we noted in Observation 5.2, k is at least 2 if $s = 1$. Since $k+\delta_1 = 4$, this means $\delta_1 \leq 2$, so we find $\delta_1 = 2$. But $k = \delta_1 = 2$ and $s_2 = 1$ contradicts (5.8).

The only remaining case is $\delta_0 = 1, s_1 = x = 0, s_2 = 2$, $k = 0$, $\delta_1 = 4$. Then from (5.5), we have $\text{OPT} = 7$ and $FT = 12$. Since $\delta_1 = 4$, there are four large singular items.

We have two singular special items and the d_0 item. None of these fit with the large singular items by Lemma 3.5. Whenever FT creates a new special bin, let the *partner item* be an item that would have been in a critical bin and is now in a regular bin. It could be that some item is a partner item twice.

Case b1. The two special items and two partner items are four distinct items. This means that the first partner item p_1 was no longer alone in a bin when the second special item arrived, because we use FF and p_1 was obviously suitable to be a partner item. Hence, p_1 was already in a 2^+-bin at that time, and $p_1 \neq d_0$. Since p_1 is not smaller than the first special item, we have now identified four items that do not fit together with any large singular item.

Match the four large singular items to these four items. Their total size is more than 4. We are left with five regular 2^+-bins, where we have matched the item p_1 from one bin B (not the first one!) with a large singular item. Item p_1 does not fit into the first bin, since we use FF. So the total size of item p_1 and the items in the first bin is more than 1. The remaining three regular bins have total size more than 2 by Lemma 2.8. Thus the total size of all the items is more than $4+1+2 = 7$, contradicting that $\text{OPT} = 7$.

Case b2. The partner item p_1 for the first special item either becomes a partner item again or becomes a special item itself.

What happened between the creation of the first and the second special bin? No item was placed in a bin with p_1, so all items were either placed in earlier bins or are large and do not fit with p_1 in a bin. No critical bin can be formed, because that would have to involve p_1.

Hence, two (unmatched) critical bins existed already when p_1 was a partner item for the first time. The second one of them could not have been created when there were already three regular bins by definition of FT. Thus, these two critical bins are among the first three regular bins. This means that the two regular 2^+-bins which appear later were never critical, so they had a packed size of at least 3/4 when they got their second item.

Now we add up all the sizes. The three singular bins with non-large items together with the four large singular items have total size more than $3m+4(1-m) = 4-m$, where $m \le 1/2$ is the smallest size of any singular non-large item. The first three regular bins have total level more than 2 by Lemma 2.8. Finally, the next two regular bins have total level at least 3/2. This means the total level of all the bins is more than $4-m+2+3/2 \ge 7$ since $m \le 1/2$, a contradiction to $\text{OPT} = 7$.

In conclusion, we have shown the following theorem.

THEOREM 5.4. *If there exists an open special bin after all items have arrived, then* $FT \le 5/3 \cdot \text{OPT}$.

6 Each special bin contains also a large item

We introduce a weight function that is a natural modification of the one that was used in the tight analysis of FF [4]. This modification ensures that every optimal bin (i.e., a bin in an optimal solution) has weight at most 5/3 (see Lemma 6.2). We no longer have that FF has a weight of 1 per bin (because it is not 5/3-competitive, after all), but we can show that the *amortized* weight of a bin of FT is at least 1.

DEFINITION 6.1. *For any item a we define its* regular weight *as* $r(a) = \frac{6}{5}a$. *We also define the* bonus *of the items that is denoted by* $b(a)$ *as follows:*

$$b(a) = \begin{cases} 0 & \text{if } 0 < a \le 1/6 \\ \frac{2}{5}(a - \frac{1}{6}) & \text{if } 1/6 < a \le 1/3 \\ 1/15 & \text{if } 1/3 < a \le 1/2 \\ 2/5 & \text{if } a > 1/2. \end{cases}$$

The weight *of the item a is defined as* $w(a) = r(a) + b(a)$.

For a set of items A and a set of bins $\mathcal{A}$, let $w(A)$ and $w(\mathcal{A})$ denote the total weight of all items in A or $\mathcal{A}$; similarly for r and b. Let $s(A)$ denote the total size of items in set A.

Note that if we have a set A of k items with sizes in $(1/6, 1/3]$, then the definition implies that its bonus is exactly $b(A) = \frac{2}{5}\left(s(A) - \frac{k}{6}\right)$. If A contains k items, each of size $\in (1/6, 1/2]$, then we get an upper bound $b(A) \le \frac{2}{5}\left(s(A) - \frac{k}{6}\right)$.

First we analyze the weight of the optimal bins, which is the easy part of the proof. This proof is what we based the definition of our weight function on.

LEMMA 6.2. *For every optimal bin A its weight* $w(A)$ *can be bounded as follows:*

(i) $w(A) \le 5/3$.

(ii) *If A contains no large item, then* $w(A) \le 7/5$.

Proof. In all cases $r(A) \le 6/5$, thus it remains to bound $b(A)$.

(i) A contains a large item. The bonus of the large item is 2/5. In addition, A contains at most 2 items larger than 1/6 of total size $y < 1/2$. If there are two such items then we have

$$b(A) \le \frac{2}{5} + \frac{2}{5}(y - \frac{2}{6}) < \frac{2}{5} + \frac{2}{5} \cdot \frac{1}{6} = \frac{7}{15}\,.$$

Otherwise if there is at most one such item then $b(A) \le 2/5 + 1/15 = 7/15$ again. In both cases $w(A) \le 6/5 + 7/15 = 5/3$.

(ii) A contains no large item. Either it contains at least 4 items with non-zero bonus, in which case their total bonus is at most

$$b(A) \le \frac{2}{5}(s(A) - \frac{4}{6}) \le \frac{2}{5} \cdot \frac{1}{3} = \frac{2}{15}\,.$$

Or else it contains at most 3 items with non-zero bonus and $b(A) \le 3/15 = 1/5$. In both cases, (ii) holds. ■

Throughout this section, we will assume (3.3) and derive a contradiction. Together with (3.3), by adding up the weight of all the optimal bins Lemma 6.2 implies that

$$w(I) \le \frac{5}{3} \cdot \text{OPT} \le FT - \frac{1}{3}\,. \tag{6.9}$$

In the following lemma, we exclude some extreme cases by a simple calculation of total volume.

LEMMA 6.3. *The following three properties hold.*

(i) *No regular* 2^+*-bin has level* $1/2$ *or smaller.*

(ii) *If* d_0 *exists, then* $d_0 > 1/3$.

(iii) *There exists at least one large singular item.*

Proof. (i) First note that the level of any special bin is more than 3/4 by Lemma 3.7 and because each special bin has a large item. Suppose to the contrary that there exists a 2^+-bin, say B_0, such that the level of B_0 is at most 1/2. Then this bin is regular. Moreover, any later regular bin is a dedicated bin since any item in these bins must be large (and there is a large dedicated item in each of these bins). Since the level of B_0 is at most 1/2, and there are at least two items in B_0, there is an item a in B_0 with size at most 1/4. It follows that the level of any earlier regular bin is bigger than 3/4.

If there is no dedicated bin, then any bin except B_0 has level above 3/4. Applying (3.3) and $\text{OPT} \geq 2$, we get

$$\text{OPT} \geq T > \frac{3}{4}(FT-1) \geq \frac{3}{4}\left(\frac{5}{3}\text{OPT} - \frac{2}{3}\right) = \text{OPT} + \frac{1}{4}(\text{OPT}-2) \geq \text{OPT}, \tag{6.10}$$

a contradiction. Thus there is at least one dedicated bin, say D_1. The total level of D_1 and B_0 is bigger than 1. We get that there are $\delta + 1 \geq 2$ bins with total level bigger than $(\delta+1)/2$, and the level of any other bin is bigger than 3/4. Then the next estimation is valid for the total size:

$$\begin{aligned}\text{OPT} \geq T &> \frac{3}{4}(FT-\delta-1) + \frac{1}{2}(\delta+1)\\ &= \frac{3}{4}FT - \frac{1}{4}\delta - \frac{1}{4}\\ &\geq \frac{5}{4}\text{OPT} + \frac{1}{4} - \frac{1}{4}\text{OPT} - \frac{1}{4} = \text{OPT},\end{aligned}$$

which is a contradiction. Here we have used $\delta \leq \text{OPT}$.

(ii) Suppose $d_0 = 1/3 - x$ with some $0 \leq x \leq 1/12$. Then the level of any other regular bin is bigger than $2/3 + x$. This holds also for the special bins, since any special bin has level now bigger than 3/4. By $FT \geq 2$ and (3.3), we get for the total size that

$$\begin{aligned}\text{OPT} \geq T &> (2/3+x)(FT-1) + (1/3-x)\\ &= \frac{2}{3}FT - \frac{1}{3} + (FT-2)x\\ &\geq \frac{2}{3}\left(\frac{5}{3}\text{OPT} + \frac{1}{3}\right) - \frac{1}{3}\\ &= \text{OPT} + \frac{1}{9}(\text{OPT}-1) \geq \text{OPT},\end{aligned}$$

a contradiction. If d_0 is even smaller, i.e. $d_0 \leq 1/4$, then the level of any other bin is bigger than 3/4, and hence we get the same contradiction as in (6.10).

(iii) Suppose there is no large singular item. If d_0 exists, we consider it together with the first special bin; their total level is above 1. The level of any other special bin is above 2/3 (in fact it is even bigger than 3/4). Consider the moment when the first special bin was created. There existed at least three interesting bins from then on (Lemma 3.4), and the bin of d_0 is not one of them. We can apply Lemma 3.6 and (3.3), and we get for the total size that

$$\begin{aligned}\text{OPT} \geq T &> \frac{2}{3}(FT-2) + 1 = \frac{2}{3}FT - \frac{1}{3}\\ &\geq \frac{2}{3}\left(\frac{5}{3}\text{OPT} + \frac{1}{3}\right) - \frac{1}{3} = \frac{10}{9}\text{OPT} - \frac{1}{9} \geq \text{OPT},\end{aligned}$$

a contradiction. ∎

For the analysis of the bins of FT, we partition them in several sets. Let $\mathcal{D}$ be the set of dedicated bins. Let $\mathcal{B}$ be the set of all 2^+-bins.

The set $\mathcal{B}$ is further partitioned into three parts:

- Let $\mathcal{S}$ be the set of special bins and their matches; note that $|\mathcal{S}| = 2s$.
- Let $\mathcal{G}$ be the set of good bins in $\mathcal{B} \setminus \mathcal{S}$.
- Let $\mathcal{C} = \mathcal{B} \setminus (\mathcal{S} \cup \mathcal{G})$. These bins are either critical bins or 3^+-bins that are not good. We number the bins in $\mathcal{C}$ according to their order in the packing of FT: $\mathcal{C} = \{C_1, \ldots, C_{|\mathcal{C}|}\}$. Let τ be the number of critical bins among $C_2, \ldots, C_{|\mathcal{C}|}$.

We start by estimating the weight of $\mathcal{D}$, $\mathcal{G}$, and $\mathcal{S}$, which are the easy cases.

Lemma 6.4. *We have $w(\mathcal{D}) - \delta > -\delta_0/3$.*

Proof. If there is no bin d_0, each bin in $\mathcal{D}$ has weight more than 1, and we are done. Else, by Lemma 6.3, $d_0 > 1/3$ and $\delta_1 \geq 1$. Now the total size of the dedicated items is greater than $\delta/2$, thus the total regular weight of these bins is greater than $3\delta/5$. The bonus of any of the δ_1 large dedicated items is 2/5, and the bonus of d_0 is 1/15. Thus the total weight is $w(\mathcal{D}) > 3\delta/5 + \frac{2}{5}(\delta-1) + 1/15 = \delta - 1/3$. ∎

Lemma 6.5. *For every bin $B \in \mathcal{G}$, we have $w(B) \geq 1$, thus $w(\mathcal{G}) - |\mathcal{G}| \geq 0$.*

Proof. If B contains a large item, or the level of the bin is at least 5/6, the weight is at least 1. Else, B has two items of combined size at least 3/4 but no large item, so the largest item has size in $[3/8, 1/2]$ and bonus 1/15 and the second largest item has size at least 1/4 and thus bonus at least 1/30. Thus $w(B) = r(B) + b(B) \geq \frac{6}{5} \cdot \frac{3}{4} + \frac{1}{15} + \frac{1}{30} = 1$. ∎

LEMMA 6.6. *Let S be a special bin with a special item a and B its match. Let b and c denote the first two items in bin B. Then the total weight of these three items is more than $4/3$. Consequently the total weight of the special bin of a and bin B is more than $7/3$ and $w(\mathcal{S}) - |\mathcal{S}| > s/3$.*

Proof. Bin B was a critical bin when a became a special item. This means that it was a 2-bin with items b and c at that point and a did not fit there by Lemma 3.7. Thus $a+b+c > 1$ and the regular weight of these three items is more than $6/5$.

We claim that the bonus of the three items is at least $2/15$. If there is a large item among them, or there are two items of size at least $1/3$ among them, the claim holds. Otherwise each item is of size at most $1/2$, and there is exactly one of size more than $1/3$. Let us denote their sizes by $x > y \geq z$. Then $1/3 < x \leq 1/2$, thus $y + z > 1/2$, and both y and z are in $[1/6, 1/3)$. For the total bonus, we get $b(x) + b(y) + b(z) = 1/15 + \frac{2}{5}(y - \frac{1}{6}) + \frac{2}{5}(z - \frac{1}{6}) = \frac{2}{5}(y+z) - \frac{1}{15} > \frac{2}{5} \cdot \frac{1}{2} - \frac{1}{15} = \frac{2}{15}$.

Thus the total weight of the three items is more than $6/5 + 2/15 = 4/3$.

Regarding the second claim we only need to recall that in any special bin there exists also a large item, and this item has weight more than 1. Hence, for every pair (S, B), the total weight of the two bins is more than $2 + 1/3$. Finally, recall that $|\mathcal{S}| = 2s$. ∎

In the previous parts, we have shown that the weight per bin is typically at least 1. One exception is the bin of d_0, in which we have less weight and this constitutes the hard case later. Another exception are the special bins in which we have $1/3$ extra weight per bin. Later it turns out that on each critical bin in $\mathcal{C}$ we have about $1/15$ too little weight. So, the next claim which relates τ, the number of critical bins in $\mathcal{C} \setminus \{C_1\}$, to s is essential in our proof and in fact gives some justification for creating special bins at regular intervals.

CLAIM 6.7. *$\tau \leq 3s$.*

Proof. There are at most $4s + 1$ critical bins and s of them are matched, and therefore contained in $\mathcal{S}$ instead of $\mathcal{C}$. The claim follows if C_1 is critical, or there are fewer than $4s+1$ critical bins. Suppose there are exactly $4s+1$ critical bins, and C_1 is not critical. Then the $(4s+1)$-th critical bin is at least the $(4s+2)$-th interesting bin, there exists an unmatched critical bin, thus instead of this critical bin a new special bin would be created, a contradiction. ∎

Now we estimate the weight of the bins in $\mathcal{C}$. We need the next amortization lemma.

LEMMA 6.8. *Let C_i and C_j be two bins in $\mathcal{C}$, $i < j$.*

(i) *If the size of C_i is at least $2/3$ and C_j is a 3^+-bin, then the regular weight of C_i plus the bonus of three items in C_j is at least 1.*

(ii) *If the size of C_i is at least $2/3$ and C_j is a 2-bin, then the regular weight of C_i plus the bonus of the two items in C_j is at least $14/15$.*

(iii) *If C_j is a 2-bin and both C_i and C_j have size at least $2/3 + \varepsilon/2$, for some $\varepsilon > 0$, then the regular weight of C_i plus the bonus of the two items in C_j is at least $14/15 + 2\varepsilon/5$.*

(iv) *If C_i has size $2/3 - \varepsilon$, for some $\varepsilon > 0$, then C_j is critical and the weight of C_j is at least $14/15 + 12\varepsilon/5$.*

Proof. (i) Since C_i is critical, or a 3^+-bin which is not good, so its level is less than $5/6$. Let its level be $5/6 - x$ for some $0 < x \leq 1/6$. Then each item in the 3^+-bin C_j is larger than $1/6 + x$, so

$$r(C_i) + b(C_j) \geq \frac{6}{5}\left(\frac{5}{6} - x\right) + 3 \cdot \frac{2}{5}x = 1.$$

(ii) Following the proof of (i), we now get

$$r(C_i) + b(C_j) \geq \frac{6}{5}\left(\frac{5}{6} - x\right) + 2 \cdot \frac{2}{5}x = 1 - \frac{2}{5}x \geq \frac{14}{15}.$$

(iii) Denote the size of C_i by $2/3 + \varepsilon_i/2$ for some $\varepsilon_i \geq \varepsilon$. Items in C_j must have size more than $1/3 - \varepsilon_i/2$. Additionally, at least one of them must have size more than $1/3$. Then

$$r(C_i) + b(C_j) \geq \frac{6}{5}\left(\frac{2}{3} + \frac{\varepsilon_i}{2}\right) + \frac{2}{5}\left(\frac{1}{6} - \frac{\varepsilon_i}{2}\right) + \frac{1}{15} = \frac{14}{15} + \frac{2\varepsilon_i}{5} \geq \frac{14}{15} + \frac{2\varepsilon}{5}.$$

(iv) C_j contains exactly two items each of size larger than $1/3 + \varepsilon$, so it is a 2-bin and therefore must be critical. Its weight is at least

$$w(C_j) = r(C_j) + b(C_j) > 12/15 + 12\varepsilon/5 + 2/15.$$

∎

LEMMA 6.9. *If $\tau = 0$, then*

$$w(\mathcal{C}) - |\mathcal{C}| \geq -\frac{2}{5}. \tag{6.11}$$

If the size of $C_{|\mathcal{C}|}$ is at least $2/3$ then

$$w(\mathcal{C}) - |\mathcal{C}| \geq \frac{-3-\tau}{15}. \tag{6.12}$$

If $\tau > 0$ and the size of $C_{|\mathcal{C}|}$ is $2/3-\varepsilon$ for $\varepsilon > 0$ then

$$w(\mathcal{C}) - |\mathcal{C}| \geq \frac{-3-\tau}{15} + \left(\frac{2}{5}\tau - \frac{8}{5}\right)\varepsilon. \tag{6.13}$$

Proof. First note, that in case $|\mathcal{C}| = 0$, set $\mathcal{C}$ is empty, so $\tau = 0$ and (6.11) holds trivially. If $|\mathcal{C}| = 1$, again $\tau = 0$ and (6.11) follows from Lemma 6.3(i) by considering the weight of the only bin in $\mathcal{C}$.

Thus let us suppose that $|\mathcal{C}| \geq 2$. To bound $w(\mathcal{C})$, we apply Lemma 6.8 for $j = 2, \ldots, |\mathcal{C}|$, typically with $i = j-1$, and we also get the regular weight of $C_{|\mathcal{C}|}$ in addition to these bounds. In all three cases, we use Lemma 6.8(i) $|\mathcal{C}|-\tau-1$ times, for all pairs of consecutive bins where the second bin is a 3^+-bin.

Case 1: C_i has size at least $2/3$ for $i = 1, \ldots, |\mathcal{C}|$. We apply Lemma 6.8(ii) τ times, and for the final bin use that $r(C_{|\mathcal{C}|}) \geq \frac{6}{5} \cdot \frac{2}{3} = \frac{4}{5}$. Thus we get

$$w(\mathcal{C}) \geq (|\mathcal{C}| - \tau - 1) + \frac{14}{15}\tau + \frac{4}{5} \quad = |\mathcal{C}| - \frac{1}{5} - \frac{1}{15}\tau,$$

proving (6.12). Note that for the special case $\tau = 0$, (6.12) is stronger than (6.11).

Case 2: C_k has size $2/3-\varepsilon$ for some $1 \leq k < |\mathcal{C}|$ and $\varepsilon > 0$. Note that k is unique, as any following bin must contain two items larger than $1/3+\varepsilon$. (We can also conclude from this that $\varepsilon < 1/6$.) We use Lemma 6.8(iv) to bound the total weight of C_j, $j > k$ (no amortization here). And finally we get the regular weight of C_k which is $4/5 - 6\varepsilon/5$. Note that the number of applications of Lemma 6.8(ii) and (iv) is $\tau \geq |\mathcal{C}| - k \geq 1$, since all bins in $\mathcal{C}$ following C_k are critical. Inequality (6.12) follows from

$$\begin{aligned} &w(\mathcal{C}) - |\mathcal{C}| \\ &\geq -\tau - 1 + \frac{14}{15}\tau + (|\mathcal{C}| - k)\frac{12}{5}\varepsilon + \frac{4}{5} - \frac{6}{5}\varepsilon \\ &\geq -\frac{\tau}{15} - \frac{1}{5} + \frac{12}{5}\varepsilon - \frac{6}{5}\varepsilon > \frac{-3-\tau}{15}\,. \end{aligned}$$

Case 3: $C_{|\mathcal{C}|}$ has size $2/3-\varepsilon$ for some $\varepsilon > 0$. (Since this is the last critical bin, it could even be that $\varepsilon > 1/6$.) Then, since $C_{|\mathcal{C}|}$ contains at least two items, it contains an item of size at most $1/3 - \varepsilon/2$ and thus C_i has level at least $2/3 + \varepsilon/2$ for $i = 1, \ldots, |\mathcal{C}| - 1$.

If $C_{|\mathcal{C}|}$ is a 2-bin, then $\tau > 0$. In this case, we use Lemma 6.8(iii) $\tau - 1$ times, namely for every pair of consecutive bins where the second bin is critical apart from the last such pair, which involves $C_{|\mathcal{C}|}$. We apply Lemma 6.8(i) $|\mathcal{C}| - \tau - 1$ times as usual, noting that $j < |\mathcal{C}|$ in every pair for which this lemma is applied. Finally, we apply Lemma 6.8(ii) once for $j = |\mathcal{C}|$. We note that $r(C_{|\mathcal{C}|}) = 4/5 - 6\varepsilon/5$, and obtain

$$\begin{aligned} &w(\mathcal{C}) \\ &\geq |\mathcal{C}| - \tau - 1 + (\tau - 1)\left(\frac{14}{15} + \frac{2}{5}\varepsilon\right) + \frac{14}{15} + \frac{4}{5} - \frac{6}{5}\varepsilon \\ &= |\mathcal{C}| - \frac{\tau}{15} - \frac{1}{5} + \left(\frac{2}{5}\tau - \frac{8}{5}\right)\varepsilon\,. \end{aligned}$$

This proves the bound (6.13).

If $C_{|\mathcal{C}|}$ is a 3^+-bin, then if $\tau > 0$, we gain $1/15$ compared to the above calculations, because we apply Lemma 6.8(i) instead of Lemma 6.8(ii) for $j = |\mathcal{C}|$.

Finally, if $\tau = 0$, we simply apply Lemma 6.8(i) $|\mathcal{C}| - 1$ times and the bound for $r(C_{|\mathcal{C}|})$ from above to get

$$w(\mathcal{C}) \geq |\mathcal{C}| - 1 + 4/5 - 6\varepsilon/5 \geq |\mathcal{C}| - 2/5.$$

■

Note that the bound (6.13) is stronger than the bound (6.12) only in the case that $\tau > 4$, it is the same if $\tau = 4$, otherwise it is weaker.

We are now ready to derive the desired contradiction to (6.9) in almost all cases.

Lemma 6.10. (i) *If $s \geq 2$ or $\delta_0 = 0$ then $w(I) - FT > -1/3$ (and consequently $FT \leq \frac{5}{3} \cdot \text{OPT}$).*

(ii) *If $s = \delta_0 = 1$ then $w(I) - FT > -7/15$.*

Proof. For $\tau = 0$, by Lemma 6.9 combined with the bounds for $\mathcal{D}$ (Lemma 6.4), $\mathcal{S}$ (Lemma 6.5) and $\mathcal{G}$ (Lemma 6.6), we obtain

$$w(I) - FT > s/3 - 2/5 - \delta_0/3 \geq -2/5.$$

Furthermore we get

$$w(I) - FT > 1/3 - 2/5 > -1/3$$

if $\delta_0 = 0$ (using that $s \geq 1$) or $s \geq 2$.

If $C_{|\mathcal{C}|}$ has size at least $2/3$ or $\tau \geq 4$, we have $w(\mathcal{C}) - |\mathcal{C}| \geq \frac{-3-\tau}{15}$ by Lemma 6.9. Together with the other bounds we get

$$\begin{aligned} w(I) - FT &> \frac{s}{3} + \frac{-3-\tau}{15} - \frac{\delta_0}{3} = \frac{5s-3-\tau}{15} - \frac{\delta_0}{3} \\ &\geq \frac{2s-3}{15} - \frac{\delta_0}{3} \geq -\frac{6}{15}\,, \end{aligned}$$

using that $\tau \leq 3s$ by Claim 6.7. Furthermore, using $\delta_0 = 0$ or $s \geq 2$ in the last inequality, we get an improved bound

$$w(I) - FT > -4/15 > -1/3.$$

If $C_{|\mathcal{C}|}$ has size $2/3 - \varepsilon$ and $0 < \tau \le 3$, we obtain, using $\varepsilon < 1/6$ (Lemma 6.3),

$$\begin{aligned} w(I) - FT &> \frac{s}{3} + \frac{-3-\tau}{15} + \left(\frac{2}{5}\tau - \frac{8}{5}\right)\varepsilon - \frac{\delta_0}{3} \\ &> \frac{5s-3-\tau}{15} + \frac{1}{15}\tau - \frac{4}{15} - \frac{\delta_0}{3} \\ &= \frac{5s-7}{15} - \frac{\delta_0}{3} \ge -\frac{7}{15}., \end{aligned}$$

Again, using $\delta_0 = 0$ or $s \ge 2$ in the last inequality, we get an improved bound

$$w(I) - FT > -2/15 > -1/3.$$

■

Lemma 6.10 shows that FT is 5/3-competitive if each special bin contains a large item, except for the single remaining case $s = \delta_0 = 1$, where we have to work a bit harder.

6.1 The case $s = 1$ and $\delta_0 = 1$ For $s = 1$, using Lemma 6.2(ii) and Lemma 6.10, we have

$$FT < w(I) + \frac{7}{15} \le \frac{5}{3}\text{OPT} + \frac{7}{15}.$$

If $\text{OPT} \not\equiv 1 \pmod 3$ then this and integrality of OPT and FT is enough to conclude that $FT \le \frac{5}{3}\text{OPT}$. The only remaining case is that every bin in the optimal packing contains a large item (otherwise the bound on $w(\text{OPT})$ is tighter), and $\text{OPT} \equiv 1 \pmod 3$.

Thus $\text{OPT} = 3k + 1$ for some $k \ge 0$, and

$$FT < \frac{5}{3}(3k+1) + \frac{7}{15} = 5k + 2 + \frac{2}{15},$$

implying that $FT \le 5k+2$. In fact, we have $FT = 5k+2$ since $(5k+1)/(3k+1) < 5/3$ for $k \ge 0$.

Using Lemma 6.3(iii), the packing of FT has the following bins:

- z regular 2^+-bins with a large item for some $z \le 3k - 1$,
- $3k + 1 - z$ dedicated bins including the bin of d_0,
- one special bin with a large item, and
- $2k$ interesting bins (regular 2^+-bins without large items).

Since $s = 1$, we have $2k \ge 3$, so $k \ge 2$, $\text{OPT} \ge 7$ and $FT \ge 12$.

Claim 6.11. (i) *The first interesting bin has level at least* $3/4$.

(ii) *If $k \ge 4$, then the first two interesting bins both have level at least* $3/4$.

Proof. (i) Suppose the first interesting bin has level less than $3/4$. Consider the other $2k - 1$ interesting bins. They contain at least $4k - 2$ items of size more than $1/4$, and there is one more in the special bin and one in d_0. These $4k$ items do not fit with the $3k + 1$ large items in the optimal packing (only one fits in each bin), contradicting that $\text{OPT} = 3k + 1$.

(ii) Now suppose any of the first two interesting bins has level smaller than $3/4$. Then the other $2k - 2$ interesting bins contain at least $4k - 4$ items greater than $1/4$, plus the special item and d_0, we have $4k - 2 > 3k + 1 = \text{OPT}$ (due to $k > 3$), so they do not fit with the large items in the optimal packing. ■

Claim 6.12. *There are at least four interesting bins with level less than* $3/4$.

Proof. Suppose to the contrary that there are at most three interesting bins with level below $3/4$. We apply Lemma 3.6 to the three interesting bins with lowest level. (The total number of these bins is $2k \ge 4$.) There are $3k + 1$ other bins (the bins with large items and the bin of d_0) with average level more than $1/2$. All other bins (the special bin and $2k - 3$ interesting bins) have level at least $3/4$. Thus we get for the total size that $\text{OPT} \ge T > 2 + \frac{1}{2}(3k + 1) + \frac{3}{4}(2k - 2) = 3k + 1 = \text{OPT}$, a contradiction. ■

Claim 6.13. *We have $k = 3$, there are six interesting bins, the first and last interesting bins have level at least $3/4$, and the remaining four have level less than* $3/4$.

Proof. By Claim 6.11(i) and Claim 6.12, there are at least five interesting bins. Since their number is $2k$, we find $k \ge 3$.

Consider the set $\mathcal{B}$ of interesting bins with level smaller than $3/4$. Each bin $B \in \mathcal{B}$ apart from possibly the first one is critical, as B may contain only items of size more than $1/4$: B can therefore not receive a large item (then its level would be more than $3/4$ as soon as it got its second item), and B can only receive two items.

Now consider the last bin $B \in \mathcal{B}$. Claim 6.12 implies that B is preceeded by at least three bins in $\mathcal{B}$, thus B as well as at least two other bins in $\mathcal{B}$ are critical. Since $s = 1$, it follows that there are at most four interesting bins before B, as otherwise a second special bin would have been created instead of packing the second item into B. To see this, note that of the two critical bins in $\mathcal{B}$ preceding B, only one is used as a match for the first special bin, and both these bins must have two items already (and hence, be critical) when B is opened, since none of these bins have large items.

Since there are not more than four interesting bins before B, Claim 6.11(ii) implies that $k \leq 3$, as otherwise B is preceded by two interesting bins with level at least $3/4$ and three bins in $\mathcal{B}$.

We conclude that $k = 3$, thus there are $2k = 6$ interesting bins. It follows that B is not the last interesting bin and thus the first and last interesting bins have level at least $3/4$, while the remaining four have level less than $3/4$. ∎

Thus in the remaining case $\text{OPT} = 3k + 1 = 10$ and FT uses $5k + 2 = 17$ bins.

Suppose that the match of the special bin is a bin with level below $3/4$. Let us count the total size of the bins. The total size of the special bin and its match is more than $3/2$. There are two interesting bins with level at least $3/4$. The total level of the other three interesting bins (level below $3/4$, and not used for the match) is more than 2 by Lemma 3.6. Finally the total level of the ten dedicated bins is more than 5. This is altogether more than $3/2 + 3/2 + 2 + 5 = 10 = \text{OPT}$, a contradiction.

If the match of the special bin is a bin with level $3/4$ or more, this must be the first interesting bin B. It cannot be the last interesting bin because the special bin is created before the sixth interesting bin is created. (Hence, B received more items after it was used as a match.) When the special bin was created, there were at least three regular 2^+-bins, including B. Any bins among these three that are not interesting must contain a large item. If there is such a bin B', then compared to the previous calculation, we lose $1/12$ because there is now only one unmatched interesting bin with level at least $3/4$, but we gain $1/6$ because Lemma 3.6 can be applied to B' (together with the other interesting bins), improving the (amortized) level guarantee of B' from $1/2$ to $2/3$.

If none of the three bins contained a large item, we find a contradiction, because in that case B was the first regular bin, and any bin following a bin with level at most $3/4$ and having level below $3/4$ itself must be a 2-bin, hence be critical if it does not contain a large item. So, B would not have been used for the matching in this case. (Recall, that we always use the last suitable critical bin for the match!)

Having shown a contradiction in all cases, we conclude the following.

THEOREM 6.14. *If every special bin has a large item after all items have been packed, then* $FT \leq 5/3 \cdot \text{OPT}$.

Combining Theorems 4.1, 5.4 and 6.14 immediately leads to our main result.

THEOREM 6.15. *The algorithm FT has absolute competitive ratio* $5/3$.

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