First Fit bin packing: A tight analysis

György Dósa[1] and Jiří Sgall[2]

1 **Department of Mathematics, University of Pannonia**
Veszprém, Hungary
`dosagy@almos.vein.hu`

2 **Computer Science Institute of Charles University**
Faculty of Mathematics and Physics
Malostranské nám. 25, CZ-11800 Praha 1, Czech Republic
`sgall@iuuk.mff.cuni.cz`

Abstract

In the bin packing problem we are given an instance consisting of a sequence of items with sizes between 0 and 1. The objective is to pack these items into the smallest possible number of bins of unit size. FirstFit algorithm packs each item into the first bin where it fits, possibly opening a new bin if the item cannot fit into any currently open bin. In early seventies it was shown that the *asymptotic* approximation ratio of FirstFit bin packing is equal to 1.7.

We prove that also the *absolute* approximation ratio for FirstFit bin packing is exactly 1.7. This means that if the optimum needs Opt bins, FirstFit always uses at most $\lfloor 1.7 \cdot \mathrm{OPT} \rfloor$ bins.

Furthermore we show *matching lower bounds* for a majority of values of Opt, i.e., we give instances on which FirstFit uses exactly $\lfloor 1.7 \cdot \mathrm{OPT} \rfloor$ bins.

Such matching upper and lower bounds were previously known only for finitely many small values of OPT. The previous published bound on the absolute approximation ratio of FirstFit was $12/7 \approx 1.7143$. Recently a bound of $101/59 \approx 1.7119$ was claimed.

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1 Introduction

Bin packing is a classical combinatorial optimization problem in which we are given an instance consisting of a sequence of items with rational sizes between 0 and 1, and the goal is to pack these items into the smallest possible number of bins of unit size. FirstFit algorithm packs each item into the first bin where it fits, possibly opening a new bin if the item does not fit into any currently open bin.

Johnson's thesis [8] on bin packing together with Graham's work on scheduling [6, 7] belong to the early influential works that started and formed the whole area of approximation algorithms. The proof that the asymptotic approximation ratio of FirstFit bin packing is 1.7 given by Ullman [13] and subsequent works by Garey et al. and Johnson et al. [5, 9] were among these first results on approximation algorithms.

In this paper, we prove that also the *absolute* approximation ratio for FirstFit bin packing is exactly 1.7. This means that if the optimum needs Opt bins, FirstFit always uses at most $\lfloor 1.7 \cdot \mathrm{Opt} \rfloor$ bins. Thus we settle this open problem after almost 40 years.

Furthermore we show matching lower bounds for a majority of values of Opt, i.e., we give instances on which FirstFit uses exactly $\lfloor 1.7 \cdot \mathrm{Opt} \rfloor$ bins. More precisely, we give

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these lower bounds for all values of $\textsc{Opt}$, except when $\textsc{Opt} \bmod 10$ equals to 0 or 3; for these two remaining cases we show a lower bound of $\lfloor 1.7 \cdot \textsc{Opt} \rfloor - 1$.

Such matching upper and lower bounds were previously known only for finitely many small values of $\textsc{Opt}$. Thus our results not only give the exact worst case for most values of $\textsc{Opt}$ (8 out of 10 residue classes), but actually even give the first infinite sequence of values of $\textsc{Opt}$ for which the exact worst-case performance of $\textsc{FirstFit}$ is known.

1.1 History and related work

The upper bound on $\textsc{FirstFit}$ was first shown by Ullman in 1971 [13]; he proved that for any instance, $\text{FF} \leq 1.7 \cdot \textsc{Opt} + 3$, where FF and $\textsc{Opt}$ denote the number of bins used by $\textsc{FirstFit}$ and the optimum, respectively. Still in seventies, the additive term was improved first in [5] to 2 and then in [4] to $\text{FF} \leq \lceil 1.7 \cdot \textsc{Opt} \rceil$; due to integrality of FF and $\textsc{Opt}$ this is equivalent to $\text{FF} \leq 1.7 \cdot \textsc{Opt} + 0.9$. Recently the additive term of the asymptotic bound was improved to $\text{FF} \leq 1.7 \cdot \textsc{Opt} + 0.7$ in [15].

The absolute approximation ratio of $\textsc{FirstFit}$ got some attention recently. A significant step towards settling the question of the absolute approximation ratio was the upper bound of 1.75 by Simchi-Levy [12]. This was improved independently by Xia and Tan [15] and Boyar et al. [1] to $12/7 \approx 1.7143$ and recently Németh claimed an upper bound of $101/59 \approx 1.7119$ [10].

For the lower bound, the early works give examples both for the asymptotic and absolute ratios. The example for the asymptotic bound gives $\text{FF} = 1.7 \cdot \textsc{Opt}$ whenever $\textsc{Opt} = 10k+1$, thus it shows that the asymptotic upper bound of 1.7 is tight, see [13, 5, 9]. For the absolute ratio, an example is given with $\text{FF} = 17$ and $\textsc{Opt} = 10$, which shows that the absolute approximation ratio cannot be better than 1.7 [5, 9]. (Also an example with $\text{FF} = 34$ and $\textsc{Opt} = 20$ is claimed, but it seems that this example has never been published.)

Johnson [8, 9] has also analyzed the First Fit Decreasing algorithm, which behaves like $\textsc{FirstFit}$ but receives the items on the input sorted from the largest one to the smallest, and proved that the asymptotic approximation ratio is equal to 11/9. Johnson's bound had an additive constant of 4; this was improved several times and finally it was shown that the additive constant is exactly 2/3 [3]. That is, $\frac{11}{9}\textsc{Opt} + \frac{2}{3}$ bins are sufficient for First Fit Decreasing, but this number of bins is actually also necessary for some instances for infinitely many values of $\textsc{Opt}$. Thus for First Fit Decreasing, the asymptotic and absolute approximation ratios are not equal. In fact, the results of [3] give the exact value of the worst case for every value of $\textsc{Opt}$ and show that the worst case absolute ratio is equal to 4/3, attained for $\textsc{Opt} = 6$ when 8 bins may be needed for First Fit Decreasing. In light of this result, it is rather surprising that for $\textsc{FirstFit}$ the asymptotic and absolute approximation ratios *are* equal and no additive term is needed.

We have mentioned only directly relevant previous work. Of course, there is much more work on bin packing, in particular there exist approximation schemes for this problem, as well as many other algorithms. We refer to the survey [2] or to the recent excellent book [14].

1.2 Main ideas of our results

Once the asymptotic bound with a small additive constant is shown, a natural approach to improve absolute upper bounds is to study fixed small values of $\textsc{Opt}$ and to exclude the possibility of a higher absolute ratio for them. Indeed, solving a few such cases necessarily improves upper bounds on the absolute ratio—but cannot give a tight result. Of course, this is still far from trivial: Even for a fixed $\textsc{Opt}$, each such problem seems to lead to a new and more extensive case analysis. Instead of joining this race of incremental results, we choose a different approach to attack arbitrarily large values of OPT directly.

The first important step is a combination of amortization and weight function analysis. To illustrate our technique, we now present a new short proof of the asymptotic ratio 1.7 for FIRSTFIT. It uses the same weight function as the traditional analysis of FIRSTFIT. To use amortization, we split the weight of each item into two parts. Identifying an item a with its size, the weight of a is its scaled size $\frac{6}{5}a$ plus the bonus $v(a)$ defined as

$$v(a) = \begin{cases} 0 & \text{if } a \leq \frac{1}{6}\,, \\ \frac{3}{5}(a - \frac{1}{6}) & \text{if } a \in \left(\frac{1}{6}, \frac{1}{3}\right)\,, \\ 0.1 & \text{if } a \in \left[\frac{1}{3}, \frac{1}{2}\right]\,, \\ 0.4 & \text{if } a > \frac{1}{2}\,. \end{cases}$$

Note that there is a discontinuity only at $a = 1/2$. For a set of items B, $v(B) = \sum_{a \in B} v(a)$ denotes the total bonus and $s(B) = \sum_{a \in B} a$ the total size.

It is easy to observe that the weight of any bin B, i.e., of any set with $s(B) \leq 1$, is at most 1.7: The scaled size of B is at most 1.2, so we only need to check that $v(B) \leq 0.5$. If B contains no item larger than 1/2, there are at most 5 items with non-zero $v(a)$ and $v(a) \leq 0.1$ for each of them. Otherwise the large item has bonus 0.4; there are at most two other items with non-zero bonus and it is easy to check that their total bonus is at most 0.1.

Consider an instance I. The previous bound implies that the weight of the whole instance $\frac{6}{5}s(I) + v(I)$ is at most $1.7 \cdot \text{OPT}$.

The key part is to show that, on average, the weight of each FIRSTFIT bin is at least 1. Lemma 1.1 implies that for almost all bins with two or more items, the first part of its weight plus the second part of the weight of the *following* such bin is at least 1.

► **Lemma 1.1.** *Let B, C be two bins in the* FIRSTFIT *packing such that $s(B) \geq 2/3$, C contains at least two items, and B is opened before C. Then $\frac{6}{5}s(B) + v(C) \geq 1$.*

Proof. Since C is after B in the FIRSTFIT packing, C contains two items c and c' that do not fit in B, i.e., $c, c' > 1 - s(B)$. If $s(B) \geq 5/6$ then the lemma follows trivially without considering $v(C)$. In the remaining case, let $x \in (0, \frac{1}{6}]$ be such that $s(C) = \frac{5}{6} - x$. Thus $c, c' > \frac{1}{6} + x$ and $v(c), v(c') > \frac{3}{5}x$. We get $\frac{6}{5}s(B) + v(c) + v(c') > \frac{6}{5}(\frac{5}{6} - x) + \frac{3}{5}x + \frac{3}{5}x = 1$. ◄

Consider any FF-bin B with a single item. If $s(B) > 1/2$, then $b(B) = 0.4$ and $\frac{6}{5}s(B) + v(B) > 1$. Furthermore, at most one FF-bin has $s(B) \leq 1/2$, by the definition of FIRSTFIT.

Now consider FF-bins with two or more items. Similarly, at most one of them has size less than 2/3: If we have one such bin, any item in any later bin is larger than 1/3 and thus any later bin with two items is larger than 2/3. Now we use Lemma 1.1 for every FIRSTFIT bin B with two or more items and $s(B) \geq 2/3$ (except for the last such bin); the bin C is chosen as the next bin with the same properties.

Summing the bounds for bins with a single item plus the bounds from Lemma 1.1 for bins with two or more items (note that each bin is used at most once as B and at most once as C), we obtain that $\frac{6}{5}s(I) + v(I) \geq \text{FF} - 3$. The additive constant 3 comes from the fact that we did not bound the weight of at most three FF-bins: (i) one bin with a single item and $s(B) \leq 1/2$, (ii) one bin with two or more items and $s(B) < 2/3$, and (iii) the last bin with two or more items. Combining this with the previous bound on the total weight, we obtain $FF - 3 \leq \frac{6}{5}s(I) + v(I) \leq 1.7 \cdot \text{OPT}$ and the asymptotic bound follows.

By itself, this simplified analysis can decrease the additive constant to 0.6 (after examining the remaining three bins in the FF packing) but cannot remove it completely. To obtain the tight bound, we need to analyze different types of bins in the FF packing quite carefully. In the typical worst case, FF packing starts by bins with five or more items of size around 1/6 or smaller, followed by OPT/2 bins with two items slightly larger than 1/3, and ends by OPT bins with a single item slightly larger than 1/2. We analyze these three types of bins separately. To handle various possible situations we slightly modify the weight function (see Definition 3.5) and the amortization lemma (see Lemma 3.8).

The most delicate part of the proof analyzes the FIRSTFIT bins containing three or four items—or rather shows that they cannot play an important role in the worst case; here it is important that the amortization uses the bonus of only two items and thus the bins with three or four items are "wasteful". In the final steps of the proof, the parity of the items of size around 1/3 comes into play: Typically they come in pairs, as described above, but for odd values of OPT one of them is missing (or is in a FIRSTFIT bin of 3 or more items), and this allows us to remove the last 0.1 of the additive term. Our analysis sketched above still leaves a few values of OPT that need to be analyzed separately. However, with our framework of the general proof, even this is relatively simple compared to the previous proofs in this area. The upper bound proof is presented in Section 3.

Similar amortization was used in [11] to analyze the Best Fit bin packing algorithm. There the situation is more complicated, as the notion of the "following bin" is not clear, in fact a careful choice is needed. Currently we are not able to fully extend our bounds to Best Fit. The bottleneck seems to be the analysis of the bins with three and four items.

For the lower bounds we modify the instance from [5, 9]. The original construction is quite intricate. Fortunately—and perhaps also surprisingly—it is sufficient to carefully analyze the high-level structure of the instance, add to it a few new jobs, and carefully position them in the input instance. See Section 4 for the details.

2 Notations

Let us fix an instance I with items $a_1, \ldots a_n$ and denote the number of bins in the FIRSTFIT and optimal solutions by FF and OPT, respectively. We will often identify an item and its size. For a set of items A, let $s(A) = \sum_{a \in A} a$, i.e., the total size of items in A and also for a set of bins $\mathcal{A}$, let $s(\mathcal{A}) = \sum_{A \in \mathcal{A}} s(A)$. Furthermore, let $S = s(I)$ be the total size of all items of I. Obviously $S \leq \text{OPT}$.

The bins in the FF packing are ordered by the time they are opened (i.e., the first item is packed into them). We refer to this order when we say that one bin is before or after another one, or when we speak about the first or last bin.

A bin is called a k-*bin* or k^+-*bin*, if it contains exactly k items or at least k items, respectively, for an integer k. An item is called k-item if FF packs it into a k-bin.

We classify the the FF bins into three groups. If a 2^+-bin B satisfies $s(B) \geq 5/6$, it is a *big* bin; $\mathcal{B}$ denotes the set of all big bins and β their number. Any other 2^+-bin C is a *common* bin; $\mathcal{C}$ denotes the set of all common bins and γ their number. Finally, any 1-bin D is a *dedicated* bin; $\mathcal{D}$ denotes the set of all dedicated bins and δ their number. The items in big, common, and dedicated bins are called B-items, C-items, and D-items, respectively. Finally, let C2-items be the items in common 2-bins. The common and dedicated bins are typically denoted by C and D, and C-items and D-items by c and d (with indices and other decorations). We use B for generic bins (typically big or common) and b for items that may be in big or common bins. If there exists a D-item with size at most 1/2, denote it d_0; otherwise d_0 is undefined. We shall see in Lemma 3.2(i) that there is at most one such item.

3 The upper bound

3.1 Preliminaries

We state a few basic properties of FF packings. Assumption 3.1 as well as all parts of Lemma 3.2 are known and easy facts used explicitly or implicitly in previous works on FirstFit including [12, 15, 1].

▶ Assumption 3.1. We assume, without loss of generality, that no two items a_i and a_j are packed into the same bin both in FF and Opt solutions.

This is w.l.o.g., since any such items may be replaced by a single item of size $a_i + a_j$ that arrives at the time of arrival of the first of the original items. It is easy to see that both FF and Opt solutions are unchanged (except for this replacement).

▶ **Lemma 3.2.** *In the* FF *packing the following holds:*

(i) *The sum of sizes of any two* FF*-bins is greater than 1. The total size of any $k \geq 2$* FF*-bins is greater than $k/2$.*

(ii) *The D-items are packed into different optimal bins. Thus $\delta \leq$* Opt*.*

(iii) *There is at most one common bin C_0 with $s(C_0) \leq 2/3$. Furthermore, if $s(C_0) = 2/3 - 2x$ for $x \geq 0$ then for any other 2^+-bin (i.e., any other common or big bin) B we have $s(B) > 2/3 + x$; in addition, if B is opened after C_0, then $s(B) > 2/3 + 4x$.*

(iv) *If $k \geq 3$, then the total size of k arbitrary 2^+-bins is greater than $\frac{2}{3}k$.*

(v) *Suppose that $k \geq 1$, we have $k+1$* FF*-bins $B_1, B_2, \ldots, B_k, B$, in this order, and such that B is a k^+-bin. Then the sum of the sizes of these $k+1$ bins is greater than k.*

Proof. **(i):** The first item in any FF-bin does not fit in any previous bin, thus the sum of their sizes is greater than 1 already at the time when the second bin is opened. For k bins, order the bins cyclically and sum the inequalities $s(B_i) + s(B_j) > 1$ for pairs of adjacent bins.

(ii): Follows from (i), as the size of each D-item equals the size of its dedicated FF-bin.

(iii): If B is after C_0, then it contains only items of size larger than $1 - s(C_0) = 1/3 + 2x$; since it contains two items, $s(B) > 2/3 + 4x$ follows. If B is before C_0, then notice that C_0 contains an item of size at most $s(C_0)/2 = 1/3 - x$; This item was not packed into B, thus it follows that $s(B) > 2/3 + x$.

(iv): Follows immediately from (iii).

(v): Let x be the minimum of $s(B_i)$, $i = 1, \ldots, k$. Then by the FF-rule, any item in bin B is larger than $1 - x$. Since there are at least k items in bin B, we have $s(B) + \sum_{i=1}^{k} s(B_i) > k(1 - x) + kx = k$. ◀

Now we assume that the instance violates the absolute ratio 1.7 and derive some easy consequences that exclude some degenerate cases. The first claim, Opt ≥ 7, follows from [1, 15]; we include its proof for completeness. Note that the values of $1.7 \cdot$ Opt are multiples of 0.1 and FF is an integer, thus FF $> 1.7 \cdot$ Opt implies FF $\geq 1.7 \cdot$ Opt $+ 0.1$. Typically we derive a contradiction with the fact $S \leq$ Opt stated above.

▶ **Lemma 3.3.** *If* FF $> 1.7 \cdot$ Opt *then the following holds:*

(i) Opt ≥ 7.

(ii) *No common bin C has size $s(C) \leq 1/2$.*

(iii) *The number of dedicated bins is bounded by $\delta \geq 3$.*

(iv) *The number of common bins is bounded by $\gamma \geq$* Opt*$/2+1 > 4$. If* FF $\geq 1.7 \cdot$Opt$+\tau/10$ *for some integer $\tau \geq 1$ then $\gamma >$ (*Opt *$+ \tau)/2$.*

Proof. **(i):** If $\textsc{Opt} \in \{3,4,5,6\}$ and $\text{FF} > 1.7 \cdot \textsc{Opt}$ then we can verify that both $\text{FF} \geq 2 \cdot \textsc{Opt} - 1$ and $\text{FF} \geq \textsc{Opt} + 3$. Using Lemma 3.2(ii), the number of 2^+-bins is $\beta + \gamma = \text{FF} - \delta \geq \text{FF} - \textsc{Opt} \geq 3$. Thus we can use Lemma 3.2(v) and obtain a contradiction:

$$S > \frac{2}{3}(\beta+\gamma) + \frac{1}{2}\delta = \frac{1}{6}(\beta+\gamma) + \frac{1}{2}\text{FF} \geq \frac{1}{6} \cdot 3 + \frac{1}{2}(2 \cdot \textsc{Opt} - 1) = \textsc{Opt}\,.$$

If $\textsc{Opt} = 2$ and $\text{FF} > 1.7 \cdot \textsc{Opt}$ then $\text{FF} \geq 4$, and by Lemma 3.2(i) we have $S > 4 \cdot \frac{1}{2} = \textsc{Opt}$, a contradiction. For $\textsc{Opt} = 1$, FirstFit is trivially optimal.

(ii): Suppose that $s(C_0) \leq 1/2$ for a contradiction. Lemma 3.2(iii) implies that any big or common bin C before C_0 has $s(C) \geq 3/4$. Furthermore, any bin after C_0 is a D-bin (as it can contain only items larger than $1/2$) and by Lemma 3.2(i), the total size of C_0 and all D-bins is at least $(\delta+1)/2$. Thus we can obtain a contradiction by using $\textsc{Opt} \geq 7$ from (i) and $\delta \leq \textsc{Opt}$ from Lemma 3.2(ii) as follows:

$$\begin{aligned} S &> \frac{3}{4}(\beta+\gamma-1) + \frac{1}{2}(\delta+1) = \frac{3}{4}\text{FF} - \frac{1}{4}(\delta+1) \\ &\geq \frac{3}{4}\left(\frac{17}{10}\textsc{Opt} + \frac{1}{10}\right) - \frac{1}{4}(\textsc{Opt}+1) = \frac{41}{40}\textsc{Opt} - \frac{7}{40} \geq \textsc{Opt}\,. \end{aligned}$$

(iii): Suppose for a contradiction that $\delta \leq 2$. Then each FF-bin contains at least two items, except for at most two dedicated FF-bins. Since $\textsc{Opt} \geq 7$ from (i), we can apply Lemma 3.2(iv) for the $\text{FF} - 2 \geq 3$ of 2^+-bins and Lemma 3.2(i) for the remaining two bins, and thus we obtain a contradiction as follows:

$$S > \frac{2}{3}(\text{FF}-2) + 1 \geq \frac{2}{3}\left(\frac{17}{10}\textsc{Opt} + \frac{1}{10} - 2\right) + 1 = \frac{17}{15}\textsc{Opt} - \frac{4}{15} > \textsc{Opt}\,.$$

(iv): To obtain the first bound from the second one, use $\tau = 1$ and the integrality of $\textsc{Opt}$. Now suppose for a contradiction that $\gamma \leq (\textsc{Opt}+\tau)/2$. If $\gamma \geq 3$, then we use Lemma 3.2(v) for $\mathcal{C}$, Lemma 3.2(i) for $\mathcal{D}$, and the fact that the remaining bins are big, and we obtain

$$\begin{aligned} S &> \frac{5}{6}(\text{FF}-\gamma-\delta) + \frac{2}{3}\gamma + \frac{1}{2}\delta = \frac{5}{6}\text{FF} - \frac{1}{6}\gamma - \frac{1}{3}\delta \\ &\geq \frac{5}{6}\left(\frac{17}{10}\textsc{Opt} + \frac{\tau}{10}\right) - \frac{\textsc{Opt}+\tau}{12} - \frac{1}{3}\textsc{Opt} = \textsc{Opt}\,, \end{aligned}$$

a contradiction. If $\gamma \leq 2$ then

$$\begin{aligned} S &> \frac{5}{6}(\text{FF}-\delta-2) + \frac{1}{2}(\delta+2) = \frac{5}{6}\text{FF} - \frac{1}{3}(\delta+2) \\ &\geq \frac{5}{6}\left(\frac{17}{10}\textsc{Opt} + \frac{\tau}{10}\right) - \frac{1}{3}\textsc{Opt} - \frac{2}{3} \geq \textsc{Opt} + \frac{\textsc{Opt}+1}{12} - \frac{2}{3} \geq \textsc{Opt}\,, \end{aligned}$$

using (i) in the last step, and we obtain a contradiction as well. ◀

3.2 The weight function and the main lemma

Now we introduce the main ingredients of our analysis: the modified weight function and the main lemma that is used for the amortized analysis of the weight of FF bins. As in the simple proof in the introduction and previous bin packing literature, our ultimate goal is to prove that each Opt-bin has weight at most 1.7 and each FF-bin has an amortized (average) weight at least 1.

It is convenient to describe the weight of each item a in two parts. The first part, $\overline{\overline{w}}(a)$, is called the regular (part of the) weight, and it is proportional to the size of a; it is the same as in the simple proof. The other part, $\overline{w}(a)$ is called the bonus and it is modified so that it depends both on the size of a and the type of FF-bin where a is packed. B-items have no bonus. C-items have bonus equal to 0 for items of size at most 1/6, equal to 0.1 for items of size at least 1/3, and linearly interpolated between these values. D-items have bonus 0.4 if they have size at least 1/2 and slightly smaller if they have smaller size (this concerns only the single item d_0).

Compared to the simple proof and the previous literature, we make several modifications to the weight function. The first two are mostly a matter of convenience and simplification of the case analysis in the proof. First, we move the bonus from the items larger than 1/2 to the D-items. Mostly these are actually the same items, except for d_0. As we shall see later, in the tight cases, each OPT-bin contains a D-item and this change allows a more uniform analysis. Second, we decrease some of the weights that we do not use in the proof, namely we do not put any bonus on B-items and decrease the bonus on d_0 (this is necessary to guarantee that its OPT-bin has weight at most 1.7; however, in tight cases d_0 is very close to 1/2). The third change is essential in our last step of the proof where we remove the remaining additive constant of 0.1. We define a set of at most two exceptional C-items whose bonus is decreased to 0. Since they are in 3^+-bins in the FF packing, this does not change the analysis of the FF packing significantly. On the other hand, the exceptional items are chosen so that, if they exist, then one OPT-bin is guaranteed to have weight at most 1.6, which is exactly the necessary improvement.

Formally we define the exceptional items as follows:

▶ **Definition 3.4.** If OPT $\equiv 7 \pmod{10}$ and there exists an OPT-bin E that contains no C2-item, then fix any such bin E for the rest of the proof. Otherwise E is undefined. If E contains at most two C-items with size larger than 1/6, denote the set of these items E'. Otherwise (if there are three or more C-items in E or no E exists) put $E' = \emptyset$.

Let us call E the *exceptional* bin and the items in E' the *exceptional* items.

Note that there is at most one exceptional item in each FF-bin by Assumption 3.1. Later we shall show that in a potential counterexample with FF $= 1.7 \cdot$ OPT$+0.1$ the bin E exists.

▶ **Definition 3.5.** The weight function is defined as follows:

For a B-item b we define $\quad \overline{w}(b) = 0.$

For a C-item c we define
$$\overline{w}(c) = \begin{cases} 0 & \text{if } c \le \frac{1}{6} \text{ or } c \in E', \\ \frac{3}{5}(c - \frac{1}{6}) & \text{if } c \in \left[\frac{1}{6}, \frac{1}{3}\right] \text{ and } c \notin E', \\ 0.1 & \text{if } c \ge \frac{1}{3} \text{ and } c \notin E'. \end{cases}$$

For a D-item d we define
$$\overline{w}(d) = \begin{cases} 0.4 & \text{if } d \ge \frac{1}{2}, \\ 0.4 - \frac{3}{5}(\frac{1}{2} - d) & \text{if } d < \frac{1}{2}. \end{cases}$$

For every item a we define $\quad \overline{\overline{w}}(a) = \frac{6}{5}a$ and $w(a) = \overline{\overline{w}}(a) + \overline{w}(a)$.

For a set of items A and a set of bins $\mathcal{A}$, let $w(A)$ and $w(\mathcal{A})$ denote the total weight of all items in A or $\mathcal{A}$; similarly for $\overline{\overline{w}}$ and $\overline{w}$. Furthermore, let $W = w(I)$ be the total weight of all items of the instance I.

In the previous definition, the function $\bar{w}$ is continuous on the case boundaries. Furthermore, if we have a set A of k C-items not from E with size in $[\frac{1}{6}, \frac{1}{3}]$, then the definition implies that the bonus of A is exactly $\overline{w}(A) = \frac{3}{5}\left(s(A) - \frac{k}{6}\right)$. More generally, if A contains at least k items, each of size at least 1/6, and no D-item, then we get an upper bound $\overline{w}(A) \le \frac{3}{5}\left(s(A) - \frac{k}{6}\right)$.

First we analyze the weight of the OPT-bins.

▶ **Lemma 3.6.** *For every optimal bin A its weight $w(A)$ can be bounded as follows:*
(i) $w(A) \le 1.7$.
(ii) *If E is the exceptional* OPT*-bin then $w(E) \le 1.6$.*
(iii) *If A contains no D-item, then $w(A) \le 1.5$.*

Proof. In all cases $\overline{\overline{w}}(A) \le 1.2$, thus it remains to bound $\overline{w}(A)$. We distinguish three cases:

Case 1: A contains no D-item. Either it contains at least 4 items with non-zero bonus, in which case their total bonus is at most $\overline{w}(A) \le \frac{3}{5}(s(A) - \frac{4}{6}) \le \frac{3}{5} \cdot \frac{2}{6} = 0.2$. Or else it contains at most 3 items with non-zero bonus and $\overline{w}(A) \le 0.3$. In both subcases (iii) follows and thus (ii) also holds if $E = A$.

Case 2: A contains a D-item larger than 1/2. The bonus of the D-item is 0.4. If $E = A$ then A has no other item with non-zero bonus and both (i) and (ii) hold. Otherwise, in addition to the D-item, A contains at most 2 items larger than 1/6 and no other items have non-zero bonus. If there is at most one such item, its bonus is at most 0.1 and (i) follows. If there are two such items, let their total size be y; note that $y < 1/2$. The bonus of A is at most $\overline{w}(A) \le 0.4 + \frac{3}{5}(y - \frac{2}{6}) < 0.4 + \frac{3}{5} \cdot \frac{1}{6} = 0.5$.

Case 3: A contains d_0. Let the size of d_0 be $\frac{1}{2} - x$ for $x \ge 0$. We have $\overline{w}(d_0) = 0.4 - \frac{3}{5}x$. We distinguish two subcases.

Case 3.1: A contains at most two items other than d_0 and larger than 1/6. Then their total size is at most $\frac{1}{2} + x$. If $E = A$ then they have no bonus and both (i) and (ii) hold. Otherwise their bonus is at most $0.1 + \frac{3}{5}x$ and (i) holds.

Case 3.2: If A contains at least three items other than d_0 and larger than 1/6. Then their total bonus is at most $\frac{3}{5}x$, thus $\overline{w}(A) \le 0.4$ and both (i) and (ii) hold. (This subcase may also happen if $E = A$, but there is no need to distinguish this in the proof.) ◀

Next we analyze the weight of FF-bins. The case of big and dedicated bins is easy:

▶ **Lemma 3.7.** (i) *The total weight of the big bins is $w(\mathcal{B}) \ge \beta$.*
(ii) *The total weight of the dedicated bins is $w(\mathcal{D}) > \delta$.*

Proof. (i): For every big bin B, $w(B) = \overline{\overline{w}}(B) = \frac{6}{5}s(B) \ge \frac{6}{5} \cdot \frac{5}{6} = 1$.

(ii): If d_0 is undefined then for every dedicated bin D, $w(D) = \frac{6}{5}s(D) + 0.4 > \frac{6}{5} \cdot \frac{1}{2} + 0.4 = 1$ and the claim follows. If d_0 exists and has size $\frac{1}{2} - x$ for $x \ge 0$, then every other D-item has size strictly larger than $\frac{1}{2} + x$. We also have $\delta \ge 3$ by Lemma 3.3(iii). Thus

$$w(\mathcal{D}) > (\delta - 1)\left(\frac{6}{5}\left(\frac{1}{2} + x\right) + 0.4\right) + \frac{6}{5}\left(\frac{1}{2} - x\right) + 0.4 - \frac{3}{5}x = \delta + \left((\delta - 1)\frac{6}{5} - \frac{6}{5} - \frac{3}{5}\right)x \ge \delta.$$

◀

Now we focus on the common FF-bins. The next lemma gives the key insight for the amortized analysis. It shows that for most common bins, the regular weight of the bin plus the bonus of the *next* common bin is at least 1. A similar method was used for the analysis of BESTFIT in [11]. For the rest of the upper bound section, number the common bins as $C_1, \ldots, C_\gamma$, in the order of their opening. The bins $C_2, \ldots, C_{\gamma-1}$ are called *inner* common bins. Note that there are some inner common bins, as $\gamma \ge 5$ by Lemma 3.3(iv).

▶ **Lemma 3.8.** *Let $i = 2, \ldots, \gamma$ be such that $s(C_{i-1}) \ge 2/3$. Then there exist two items $c, c' \in C_i \setminus E'$ and for any such items*

$$\overline{\overline{w}}(C_{i-1}) + \overline{w}(c) + \overline{w}(c') \ge 1\,.$$

Thus we have $\overline{\overline{w}}(C_{i-1}) + \overline{w}(C_i) \ge 1$.

Proof. If C_i is a 2-bin, then it contains no exceptional item. If C_i is a 3^+-bin, then it contains at most one exceptional item by Assumption 3.1. In both cases c and c' exist. Since C_{i-1} is common, the size of this bin is smaller than 5/6 and it is at least 2/3 by the assumption of the lemma. Let $x \in (0, \frac{1}{6}]$ be such that $s(C_{i-1}) = \frac{5}{6} - x$. Thus $c, c' > \frac{1}{6} + x$ and $\overline{w}(c), \overline{w}(c') > \frac{3}{5}x$. We get $\overline{\overline{w}}(C_{i-1}) + \overline{w}(c) + \overline{w}(c') > \frac{6}{5}(\frac{5}{6} - x) + \frac{3}{5}x + \frac{3}{5}x = 1$. ◀

3.3 The last common bin is large

The outline of the rest of the proof is this: We prove that the common FF-bins have total weight at least $\gamma - 0.2$. This part of analysis is considerably harder in case when the last common bin is smaller than 2/3, and we omit that part in this conference version. Then, since the total weight of the dedicated bins is strictly greater than δ, this implies $W > \mathrm{FF} - 0.2$. Together with $W \le 1.7 \cdot \textsc{Opt}$ now $\mathrm{FF} \le 1.7 \cdot \textsc{Opt} + 0.1$ follows. However, $\mathrm{FF} = 1.7 \cdot \textsc{Opt} + 0.1$ can hold only if $\textsc{Opt} \equiv 7 \pmod{10}$. Then we show that the exceptional bin is defined, thus $W \le 1.7 \cdot \textsc{Opt} - 0.1$ and we save the last 0.1.

▶ **Lemma 3.9.** *If $s(C_\gamma) \ge 2/3$, then the total weight of the common bins is $w(\mathcal{C}) \ge \gamma - 0.2$.*

Proof. First consider the case when every common bin has size at least 2/3. We apply Lemma 3.8 for every $i = 2, \ldots, \gamma$. The regular weight of the last bin is at least $\overline{\overline{w}}(C_\gamma) \ge \frac{6}{5} \cdot \frac{2}{3} = 0.8$. Summing all of these inequalities we obtain

$$w(\mathcal{C}) = \sum_{i=1}^{\gamma} w(C_i) \ge \overline{\overline{w}}(C_\gamma) + \sum_{i=2}^{\gamma} (\overline{\overline{w}}(C_{i-1}) + \overline{w}(C_i)) \ge 0.8 + (\gamma - 1) = \gamma - 0.2.$$

Now suppose that $s(C_k) = 2/3 - x$ for $x > 0$ and $k < \gamma$. Using Lemma 3.2(iii), each C_j, $j > k$, contains (exactly) two items larger than $1/3 + x$. Thus $\overline{w}(C_j) = 0.2$ and also $s(C_j) > 2/3 + 2x$ which implies $\sum_{i=k}^{\gamma} s(C_i) > (\gamma + 1 - k)\frac{2}{3}$. Combining these we have $\overline{\overline{w}}(C_k) + \sum_{j=k+1}^{\gamma} w(C_j) \ge (\gamma + 1 - k) - 0.2$. Adding the last inequality and the inequalities $\overline{\overline{w}}(C_{i-1}) + \overline{w}(C_i) \ge 1$ from Lemma 3.8 for $i = 2, \ldots, k$, the lemma follows. ◀

▶ **Lemma 3.10.** *Suppose $w(\mathcal{C}) \ge \gamma - 0.2$. Then*
(i) $\mathrm{FF} \le 1.7 \cdot \textsc{Opt} + 0.1$, *and*
(ii) *if the exceptional bin E is defined, then* $\mathrm{FF} \le 1.7 \cdot \textsc{Opt}$.

Proof. By Lemma 3.7 and the assumption we have $W > \beta + (\gamma - 0.2) + \delta = \mathrm{FF} - 0.2$. By Lemma 3.6(i) we have $W \le 1.7 \cdot \textsc{Opt}$. Thus $\mathrm{FF} - 0.2 < W \le 1.7 \cdot \textsc{Opt}$. Since FF and Opt are integers, (i) follows. If E is defined then by Lemma 3.6(i) and (ii) we have $W \le 1.7 \cdot \textsc{Opt} - 0.1$. Thus $\mathrm{FF} - 0.2 < W \le 1.7 \cdot \textsc{Opt} - 0.1$ and (ii) follows. ◀

To decrease the bound by the last one tenth, we only need to show that the exceptional Opt-bin is defined. First yet another auxiliary lemma:

▶ **Lemma 3.11.** *Suppose that every* Opt*-bin contains a D-item. Then no* Opt *bin contains two* 2*-items c_1 and c_2.*

Proof. For contradiction, assume we have such c_1 and c_2 and number them so that the FF-bin of c_1 is before the FF-bin of c_2. (Note that by Assumption 3.1, c_1 and c_2 are not in the same FF-bin.) Let c_3 be the other item in the FF-bin of c_1. Since c_2 was not packed into this bin, which contains only c_1 and c_3, we have $c_1 + c_2 + c_3 > 1$. This implies that c_3 cannot be in the Opt-bin of c_1 and c_2. Every Opt-bin contains a D-item by the assumption; let d_1 be the D-item in the Opt-bin of c_1 and c_2 and d_3 the D-item in the Opt-bin of c_3. By Lemma 3.2(i), $d_1 + d_3 > 1$ and thus $c_1 + c_2 + c_3 + d_1 + d_3 > 2$. As all these items are in two Opt-bins, this is a contradiction. ◀

▶ Proposition 3.12. Suppose that $s(\mathcal{C}) \geq \gamma - 0.2$. Then $\mathrm{FF} \leq 1.7 \cdot \mathrm{OPT}$.

Proof. By Lemma 3.10(i) we have $\mathrm{FF} \leq 1.7 \cdot \mathrm{OPT} + 0.1$. If $\mathrm{OPT} \not\equiv 7 \pmod{10}$ then by checking all the other residue classes we can verify that $1.7 \cdot \mathrm{OPT} + 0.1$ is non-integral. Thus $\mathrm{FF} \leq 1.7 \cdot \mathrm{OPT} + 0.1$ implies $\mathrm{FF} \leq 1.7 \cdot \mathrm{OPT}$ and we are done.

It remains to handle the case when $\mathrm{OPT} \equiv 7 \pmod{10}$ and $\mathrm{FF} = 1.7 \cdot \mathrm{OPT} + 0.1$.

First we claim that every OPT-bin contains a D-item and thus $\delta = \mathrm{OPT}$. If some OPT-bin does not contain a D-item, its weight is at most 1.5 by Lemma 3.6(iii). Thus $W \leq 1.7 \cdot \mathrm{OPT} - 0.2$. Since $\mathrm{FF} > W - 0.2$, we obtain $\mathrm{FF} \leq 1.7 \cdot \mathrm{OPT}$, a contradiction.

Lemma 3.11 now implies that no OPT-bin contains two C2-items. Note that OPT is odd, as $\mathrm{OPT} \equiv 7 \pmod{10}$. On the other hand, the number of C2-items is even (in any FF-bin there are either zero or two C2-items). Thus some OPT-bin contains no C2-item. This bin satisfies all the conditions of Definition 3.4 of the exceptional bin. Thus E is defined and by Lemma 3.10(ii) the proposition follows. ◀

Together with the omitted case of $s(\mathcal{C}) < \gamma - 0.2$, we obtain our main result.

▶ **Theorem 3.13.** *For any instance of bin packing,* $\mathrm{FF} \leq 1.7 \cdot \mathrm{OPT}$.

4 Lower bounds

To prove the lower bounds, we use the classical lower bound construction from [5, 9]. We have an input instance L with three regions of items. In the first region there are items of size close to 1/6, in the second region come items close to 1/3, and in the third region there are items with the equal size $1/2 + \delta$, for a small $\delta > 0$. We will not modify the items in this list, only add some new items before or after L, and also in between the three regions of L. Thus we need to review the properties of L with the focus on the resulting FF packing in each region; the details within each region are somewhat delicate but fortunately we can use that part as a black box. We formulate the properties of L in the next lemma, before giving our lower bound in Theorem 4.2.

▶ **Lemma 4.1** ([5, 9]). *For every k and a sufficiently small $\delta > 0$ there exists an instance L of $30k$ items such that $\mathrm{OPT} = 10k + 1$ and $\mathrm{FF} = 17k$ for L. Furthermore the following holds for $\varepsilon = 46 \cdot 18^{k-1}\delta = O(\delta)$:*

(i) *The first $10k$ items of L have size at least $1/6 - \epsilon$ and are packed into the first $2k$ FF-bins; no further item is packed later into these bins. Each of these $2k$ FF-bins is a big 5-bin, and has size at least $5/6 + \delta$;*

(ii) *The next $10k$ items of L have size at least $1/3 - \epsilon$ and are packed into the next $5k$ FF-bins; no further item is packed later into these bins. Each of these FF-bins is a common 2-bin and has size at least $2/3 + 2\delta$.*

(iii) *The last $10k$ items of L have size exactly $1/2 + \delta$ are packed into the next $10k$ FF-bins. Each of these FF-bins is a dedicated bin and has size exactly $1/2 + \delta$.*

(iv) *Moreover, all items of L, except three items, fit into $10k - 1$ bins, each of size $1 - O(\delta)$. The three remaining items have sizes $1/3 + \varepsilon$, $1/6 - 3\delta$, and $1/2 + \delta$.*

▶ **Theorem 4.2.** *For all integers $k \geq 1$ and $0 \leq i \leq 9$, there exists an instance I such that $\mathrm{OPT} = 10k + i$ and the lower bound in the top row of the next table holds. The bottom row of the table gives the upper bounds from Theorem 3.13 for a comparison.*

$i =$	0	1	2	3	4	5	6	7	8	9
$\mathrm{FF} \geq 17k +$	-1	1	3	4	6	8	10	11	13	15
$\mathrm{FF} \leq \lfloor 17k + 1.7i \rfloor = 17k +$	0	1	3	5	6	8	10	11	13	15

Furthermore, for $i = 1, \ldots, 9$ there exist instances with $\text{OPT} = i$ *and* $\text{FF} = \lfloor 1.7 \cdot i \rfloor$.

Proof. We show how the instance L from Lemma 4.1 can be modified to prove the theorem. We only show in each case that $\text{OPT} \le 10k + i$. However, equality follows as the lower bound on FF is always larger than the upper bound on FF for $\text{OPT} - 1$ (see the table in the theorem).

For $i = 0$, we create I by deleting one item of size $1/2 + \delta$ from L. Then $\text{FF} = 17k - 1$. Optimum uses $10k - 1$ bins as in Lemma 4.1(iv). Only two items of sizes $1/3 + \varepsilon$ and $1/6 - 3\delta$ remain and they are packed into the last bin, thus obtaining $\text{OPT} \le 10k$. For $k = 1, 2$, better examples with $\text{FF} = 17k$ and $\text{OPT} = 10k$ exist [5, 9], no such examples are known for $k > 2$.

For $i \ge 1$ and $k \ge 1$, we modify L by inserting new items. First we describe an optimal packing with $10k+i$ optimal bins together with the new items. The first $10k-1$ bins contain the same items as in Lemma 4.1(iv). The $(10k)$th bin contains two of the remaining items from L, namely $1/2 + \delta$ and $1/6 - 3\delta$ and a new item $c_0 = 1/3 + 2\delta$. The $(10k + 1)$st bin contains the last remaining item from L, namely $1/3 + \varepsilon$, and two new items $d_0 = 1/2 + \delta/4$ and $b_0 = 1/6 - \delta/4 - \varepsilon$. If $i > 1$, then the $(10k+j)$th bin of the optimal packing, $j = 2, \ldots, i$, contains three new items $d_j = 1/2 + \delta/4$, $c_j = 1/3 + \delta/4$ and $b_j = 1/6 - \delta/2$.

The items b_j, c_j and d_j are called B-items, C-items and D-items, respectively; they are typically packed into big, common and dedicated bins of the optimum. We have exactly i new items of each type.

Now we describe the new instance I, together with the FF packing. The instance I consists of L and some of the new items. In some cases we do not need all new items. Then we remove the remaining new items; this can obviously only decrease the optimum, thus $\text{OPT} \le 10k + i$.

All the new D-items are put at the end of L. Lemma 4.1 implies that they do not fit into any previous FF bin and thus FF puts each of them into a new dedicated bin. Furthermore, $2\lfloor i/2 \rfloor$ smallest new C-items are inserted in between the C-items and D-items in L. Since there is an even number of these new C-items and they do not fit into any of the previous bins, in FF packing they are put into $\lfloor i/2 \rfloor$ C-bins. Note that no D-item, old or new, does not fit into these new bins.

At this point we have created $\lfloor 3i/2 \rfloor$ new bins in the FF packing. Comparing this value with the table in the theorem, we have sufficiently many FF-bins for $i = 1, 2, 3, 4$, while for $i = 5, 6, 7, 8$ we need one additional FF-bin and for $i = 9$ two additional FF-bins. To create these bins, we have available all i new B-items and for odd i also one C-item, namely c_0, which is the largest one. We distinguish a few cases.

Case $i = 1, 2, 3, 4$: We discard all the remaining new items.

Case $i = 5$: We put one new C-item and four new B-items in front of L. They fit into a bin, thus FF packs them into the first bin and no other item fits in it. More precisely, the size of this bin is $1 - O(\delta)$, thus for a sufficiently small δ, no other item fits into it, as all the items have size at least $1/6 - O(\delta)$. The remaining B-item is discarded.

Case $i = 6, 7, 8$: We put 6 new B-items at the beginning of the list. They are packed in the first FF-bin and no other item will fit into it. The remaining items are discarded for $i = 7, 8$.

Case $i = 9$: We put 6 new B-items including b_0 at the beginning of the list. Again, they are packed in the first FF-bin and no other item will fit into it. We also insert $c_0 = 1/3 + 2\delta$, and the three remaining new B-items of size $1/6 - \delta/2$ between the B-items and C-items of L. None of these items fits in the previous bins, as those have size at least $5/6 + \delta$ by Lemma 4.1(i). Thus they are packed into one FF-bin of size about $5/6$. Since all the following items have size at least $1/3 - O(\delta)$, for a sufficiently small δ no further item fits into this bin. Thus this will be the second additional bin.

This completes the proof for $\textsc{Opt} \geq 10$. For $\textsc{Opt} \leq 9$, let $1 \leq i \leq 9$. Then I contains i items of each of the three sizes $1/6 - 2\delta$, $1/3 + \delta$, and $1/2 + \delta$. The items are ordered by non-decreasing size. It is easy to verify that for all $i = 1, \ldots, 9$, we have $\text{FF} = \lfloor 1.7 \cdot i \rfloor$ and also $\textsc{Opt} = i$, as we can pack into each bin three items of different sizes. ◀

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A The last common bin is small

Suppose that the size of the last common bin is smaller than 2/3. For the rest of the upper bound proof, fix $x > 0$ so that $s(C_\gamma) = \frac{2}{3} - 2x$. Lemma 3.3(ii) implies $s(C_\gamma) > 1/2$ and thus $x < 1/12$.

Since now the regular weight of the last bin is smaller than 0.8, we need to compensate for this. This is indeed possible due to the fact that now Lemma 3.2(iii) implies that the inner common bins are larger than $2/3 + x$ and this allows us to improve the bounds of Lemma 3.8 by an amount proportional to x.

Note that C_i, $i > 1$, cannot be a 5^+-bin: Since $s(C_1) < 5/6$, all items in C_i have size larger than 1/6 and five of them would add up to more than 5/6, contradicting the assumption that C_i is a common bin.

▶ **Lemma 1.1.** *For $i = 2, \ldots, \gamma - 1$ we have the following bounds:*
If C_i is a 2-bin or a 3-bin, then $\overline{\overline{w}}(C_{i-1}) + \overline{w}(C_i) \geq 1 + \frac{3}{5}x$.
If C_i is a 4-bin, then $\overline{\overline{w}}(C_{i-1}) + \overline{w}(C_i) \geq 1 + \frac{3}{10}x$.

Proof. Let y be such that $s(C_{i-1}) = \frac{5}{6} - y$. Since C_{i-1} is a common bin, $y > 0$. On the other hand, by Lemma 3.2(iii) the size of C_{i-1} is greater than $\frac{2}{3} + x$ and thus also $y < \frac{1}{6} - x$. Note that $\overline{\overline{w}}(C_{i-1}) = \frac{6}{5}(\frac{5}{6} - y) = 1 - \frac{6}{5}y$ and that every item $c \in C_i$ satisfies $c > \frac{1}{6} + y$.

Case 1: C_i is a 2-bin. Then C_i contains at least one item c of size larger than 1/3 as otherwise $s(C_{i-1}) \leq 2/3$ contradicting Lemma 3.2(iii) together with the assumption that $S(C_\gamma) < 2/3$. The other item c' in C_i satisfies $c' > \frac{1}{6} + y$. Thus

$$\overline{\overline{w}}(C_{i-1}) + \overline{w}(C_i) \geq 1 - \frac{6}{5}y + \frac{3}{5}y + 0.1 = 1.1 - \frac{3}{5}y \geq 1.1 - \frac{3}{5}\left(\frac{1}{6} - x\right) = 1 + \frac{3}{5}x\,.$$

Case 2: C_i is a 3-bin. Suppose that C_i contains an item $c > 1/3$. Then the remaining two items in C_i have size at least $\frac{1}{6} + y$ and we obtain

$$\overline{\overline{w}}(C_{i-1}) + \overline{w}(C_i) \geq 1 - \frac{6}{5}y + \frac{3}{5}(y + y) + 0.1 = 1.1 \geq 1 + \frac{3}{5}x$$

since $x < 1/12$. Otherwise all three items in C_i have size at most 1/3. We claim that one of them has size at least $\frac{1}{6} + x$, as otherwise, using $x < 1/12$, we have $s(C_i) < 3(\frac{1}{6} + x) = \frac{1}{2} + 3x < \frac{2}{3} + x$, contradicting Lemma 3.2(iii). Now we get

$$\overline{\overline{w}}(C_{i-1}) + \overline{w}(C_i) \geq 1 - \frac{6}{5}y + \frac{3}{5}(y + y + x) = 1 + \frac{3}{5}x\,.$$

Case 3: Suppose C_i is a 4-bin. All items in C_i are small, as otherwise $s(C_i) \geq \frac{1}{3} + 3 \cdot \frac{1}{6} = \frac{5}{6}$, contradicting the assumption that C_i is a common bin. As $s(C_i) > \frac{2}{3} + x$ by Lemma 3.2(iii), there must be two items with total size at least $\frac{1}{3} + \frac{x}{2}$ and their total bonus is at least $\frac{3}{5} \cdot \frac{x}{2}$. Thus

$$\overline{\overline{w}}(C_{i-1}) + \overline{w}(C_i) \geq 1 - \frac{6}{5}y + \frac{3}{5}\left(y + y + \frac{x}{2}\right) = 1 + \frac{3}{10}x\,.$$

◀

Let γ_k denote the number of k-bins that do not contain an exceptional item among the inner common bins, i.e., among $C_2, \ldots, C_{\gamma-1}$. Let $\alpha = 2(\gamma_2 + \gamma_3) + \gamma_4$.

▶ **Lemma 1.2.** *Suppose that $s(C_\gamma) < 2/3$. The following holds:*

(i) *If* $\alpha \geq 8$ *then the total weight of the common bins is at least* $w(\mathcal{C}) \geq \gamma - 0.2$.
(ii) *If* $\alpha \geq 4$ *then the total weight of the common bins is at least* $w(\mathcal{C}) \geq \gamma - 0.3$.

Proof. We apply Lemma 1.1 for any $i = 1, \ldots, \gamma - 2$ such that C_{i+1} does not contain an exceptional item. Otherwise, i.e., if C_{i+1} contains an exceptional item and also for $i = \gamma - 1$ we apply Lemma 3.8. Summing all the resulting bounds on $\overline{\overline{w}}(C_i) + \overline{w}(C_{i+1})$ and $\overline{\overline{w}}(C_\gamma) = 0.8 - \frac{12}{5}x$ we obtain that the total weight of the common bins is

$$s(\mathcal{C}) \geq \gamma - 1 + (\gamma_2 + \gamma_3)\frac{3}{5}x + \gamma_4\frac{3}{10}x + 0.8 - \frac{12}{5}x = \gamma - 0.2 + \frac{(3\alpha - 24)x}{10}.$$

For $\alpha \geq 8$ we have $3\alpha \geq 24$ and (i) follows.

For $\alpha \geq 4$ we use $x < 1/12$, which gives $(3\alpha - 24)x \geq -12x \geq -1$ and (ii) follows. ◀

▶ **Theorem 1.3** (Theorem 3.13). *For any instance of bin packing,* $\text{FF} \leq 1.7 \cdot \text{OPT}$.

Proof. If $s(C_\gamma) \geq 2/3$ then the theorem follows by Proposition 3.12. Thus assume $s(C_\gamma) < \frac{2}{3}$ and $\text{FF} \geq 1.7 \cdot \text{OPT} + 0.1$. We distinguish several cases and in each we derive a contradiction or prove the theorem statement $\text{FF} \leq 1.7 \cdot \text{OPT}$, leading to an indirect proof as well.

Case 1: $\text{OPT} \geq 21$. By Lemma 3.3(iv) we have $\gamma \geq 12$. Thus there are at least 10 inner common bins and at most 2 of them have an exceptional item. Thus $\alpha \geq \gamma_2 + \gamma_3 + \gamma_4 \geq 8$ and $w(C) \geq \gamma - 0.2$ by Lemma 1.2(i). Now Lemma 1.2(i) and Proposition 3.12 imply the theorem.

Case 2: $\text{OPT} \geq 8$, $\text{OPT} \not\equiv 4 \pmod{10}$, and $\text{OPT} \not\equiv 7 \pmod{10}$. Then $\text{FF} \geq 1.7 \cdot \text{OPT} + 0.3$, thus we can use Lemma 3.3(iv) with $\tau = 3$ and we obtain $\gamma \geq 6$. There are no exceptional items, since $\text{OPT} \not\equiv 7 \pmod{10}$, and thus $\alpha \geq 4$. Lemma 1.2(ii) implies $W > \beta + (\gamma - 0.3) + \delta = \text{FF} - 0.3 \geq 1.7 \cdot \text{OPT}$, a contradiction.

Case 3: $\text{OPT} = 14$. Then $\text{FF} = 24$. Lemma 3.3(iv) with $\tau = 2$ implies $\gamma \geq 9$. There are no exceptional items, thus $\gamma_2 + \gamma_3 + \gamma_4 \geq 7$. If $\gamma_2 + \gamma_3 \geq 1$ then $\alpha \geq 8$ and the theorem follows by Lemma 1.2(i). In the remaining case $\gamma_4 \geq 7$, thus there are seven 4-bins among the common bins. Using Lemma 3.2(v) for five of these common 4-bins, Lemma 3.2(iv) for some four of the remaining $\gamma - 5 \geq 4$ common bins and Lemma 3.2(i) for the remaining $24 - 9 = 15$ bins we get

$$S > 4 + 4 \cdot \frac{2}{3} + 15 \cdot \frac{1}{2} > 14 = \text{OPT},$$

a contradiction.

Case 4: $\text{OPT} = 17$. Then $\text{FF} = 29$. Lemma 3.3(iv) gives $\gamma \geq 10$. Thus $\gamma_2 + \gamma_3 + \gamma_4 \geq 6$. If $\gamma_4 \leq 4$ then $\alpha \geq 2(6 - \gamma_4) + \gamma_4 \geq 8$ and the theorem follows by Lemma 1.2(i). Otherwise there are five 4-bins among the common bins. Using Lemma 3.2(v) for these five 4-bins, Lemma 3.2(iv) for some five of the remaining $\gamma - 5 \geq 5$ common bins and Lemma 3.2(i) for the remaining $29 - 10 = 19$ bins we get

$$S > 4 + 5 \cdot \frac{2}{3} + 19 \cdot \frac{1}{2} > 17 = \text{OPT},$$

a contradiction.

Case 5: $\text{OPT} = 7$. Then $\text{FF} = 12$.

First we claim that $\delta = 7$. Otherwise $S > 6 \cdot \frac{2}{3} + 6 \cdot \frac{1}{2} = 7$, a contradiction.

It follows that there are at most three 2-bins in the FF packing, since by Lemma 3.11 no OPT-bin can contain two 2-items.

Next we claim that no two FF-bins have total size greater than or equal to 3/2. Otherwise there remain at least three 2^+-bins in the FF packing and $S > \frac{3}{2} + 3 \cdot \frac{2}{3} + 7 \cdot \frac{1}{2} = 7$, a contradiction.

Since there are five 2^+-bins and at most three 2-bins, there have to be at least two 3^+-bins. Let C be the last 3^+-bin and B some bin before it. Then C contains three items of size larger than $1 - s(B)$ and $s(B) + s(C) \geq s(B) + 3(1 - s(B)) = 3 - 2 \cdot s(B)$. Since no two bins have total size 3/2 or more, this implies $s(B) \geq 3/4$. Furthermore, this implies that there is a single bin before C, as otherwise there would again be two bins with total size at least 3/2. I.e., B is the first bin, C is the second bin and there are exactly three 2-bins.

C has at least three items and they are packed into different OPT-bins by Assumption 3.1. We claim that one of these three bins contains both a 2-item c and a D-item d with size $d > 1/2$: Each OPT-bin contains a D-item and there is at most one D item of size at most 1/2; furthermore, there is at most one OPT-bin not containing a 2-item, as there are six 2-items in the three 2-bins. Thus the condition excludes at most two OPT-bins. Fix c' to be an item from C packed with such a c and d in the same OPT-bin. Note that c and d are in later FF-bins than C, as B and C are the first bins and they are 3^+-bins.

We have $c' + c < 1/2$ as they are packed with $d > 1/2$ in an OPT-bin. On the other hand we claim that $s(C) - c' < 1/2$: otherwise we note that $c' > 1 - s(B)$, as c' was not packed in B and thus $s(B) + s(C) > s(B) + c' + 1/2 > s(B) + (1 - s(B)) + 1/2 = 3/2$, contradicting the first claim in the proof. Thus $s(C) + c = (s(C) - c') + (c' + c) < 1/2 + 1/2 = 1$ and FF should have packed c into C, which is the final contradiction. ◀

We note that the last case of OPT $= 7$ is also covered in the manuscript [10], we have included it for completeness.