RESEARCH STATEMENT

SAMUEL BRAUNFELD

1. Introduction

My research lies in the intersection of model theory, combinatorics, and permutation group theory. It is largely concerned with when an infinite limit structure can be assigned to a family of finite structures, and how the limit's model-theoretic properties or its automorphism group's dynamical properties are reflected in the combinatorial properties of the finite structures, and vice versa

A centerpiece of the field is the Fraïssé construction, assigning a *homogeneous* limit structure, with a very rich automorphism group, to a family of finite structures with the *amalgamation property*. Examples of homogeneous structures include the rational order $(\mathbb{Q}, <)$ (the Fraïssé limit of all finite linear orders) and the infinite random graph (the Fraïssé limit of all finite graphs). The Fraïssé construction and generalizations have been used to produce structures with prescribed properties, providing solutions to major questions in model theory, as well as other fields including permutation groups, combinatorics, and topological dynamics.

For example, the Ramsey property is a widely studied combinatorial property, and the following seminal result from [20] connects the Ramsey property for a family of finite structures to an unusually strong dynamical property of the automorphism group of its Fraïssé limit.

Theorem 1.1 (Kechris, Pestov, Todorčević)**.** *Let $G = Aut(M)$ for M homogeneous. Then G is* extremely amenable *iff the class of finite substructures of M has the* Ramsey property*.*

In addition to the Ramsey property, various other combinatorial properties have been studied that relate to whether $Aut(M)$ is amenable, $Aut(M)$ is simple, allow for the computation of the universal minimal flow of $Aut(M)$, or determine whether M can be recovered from $Aut(M)$.

Model-theoretic analysis of infinite structures can also help identify and analyze classes of combinatorial structures that are tame by some measure, such as growth rate, i.e. the number of structures of size n it contains. Studying growth rates is a common combinatorial problem, and there is particular interest in the gaps that occur in the spectrum of possible growth rates. In the following theorem for unlabeled enumeration, which confirms some longstanding conjectures of Peter Cameron and Dugald Macpherson [23, 24], the gaps were found by studying the parenthetical model-theoretic property in the limit structure M.

Theorem 1.2 ([7], B.)**.** *Let M be a countable homogeneous structure, $\mathcal{C}$ be the hereditary class of unlabeled substructures of M, and $\mathcal{C}_n$ the elements of $\mathcal{C}$ of size n. If $|\mathcal{C}_n| = o\left(\frac{\phi^n}{poly(n)}\right)$ for ϕ the golden ratio, then $|\mathcal{C}_n| = o(c^n)$ for every $c > 1$. Furthermore, one of the following holds.*

1. *(Cellularity) There are $c > 0$, $k \in \mathbb{N}$ such that $|\mathcal{C}_n| \sim cn^k$.*
2. *(Monadic stability, Morley rank 2) There are $c > 0$, $k \in \mathbb{N}$ such that $|\mathcal{C}_n| = e^{\left(\Theta\left(n^{1-\frac{1}{k}}\right)\right)}$.*
3. *(Monadic stability, Morley rank $r+2$) Let $\log^r(n)$ denote the r-fold iterated logarithm. There are $c > 0$ and $k, r \in \mathbb{N}$ such that $|\mathcal{C}_n| = e^{\left(\Theta\left(\frac{n}{(\log^r(n))^{1/k}}\right)\right)}$.*

In the best case, model-theoretic analysis gives the desired combinatorial results, and the process of proving this spurs a deeper understanding the of model-theoretic concepts or suggests new ones. Extending Theorem 1.2 has lead to renewed interest in the model-theoretic property of monadic NIP (closely related to bounded VC-dimension), and to the following conjecture connecting it to asymptotic enumeration, structural graph theory, algorithmic complexity, and monadic second-order logic.

Conjecture 1. *Let M be a countable homogeneous structure, $\mathcal{C}$ be the hereditary class of unlabeled substructures of M, and $\mathcal{C}_n$ the elements of $\mathcal{C}$ of size n. Then the following are equivalent.*

1. *$|\mathcal{C}_n| = o(c^n)$ for some $c \in \mathbb{R}$.*
2. *M is monadically NIP (i.e. every expansion of M by unary relations is NIP).*
3. *$\mathcal{C}$ is well-quasi-ordered by the substructure relation.*
4. *$\mathcal{C}$ has finite partition-width (a generalization of finite clique-width [3]).*
5. *First-order model checking in $\mathcal{C}$ is fixed-parameter tractable.*
6. *M has a decidable monadic second-order theory.*

We note that in arbitrary hereditary graph classes, these notions are known to not be so cleanly related [1, 16, 22], but our assumptions make $\mathcal{C}$ much more tame.

2. Model theory and tame hereditary classes

For a hereditary class of relational structures, i.e. one closed under taking substructures, there are many properties by which the class may be considered tame. These include slow growth rate, well-quasi order (i.e. containing no infinite antichain), some notion of bounded width, admitting fast algorithms, and ultimately a description of how the structures in the class can be constructed from simpler ones. In many cases, several of these properties occur for classes of interest, such as in the graph minor program of Robertson-Seymour, in nowhere-dense monotone graph classes [25], and in ordered graph classes of bounded twin-width [4, 29].

My aim is to characterize which classes are considered tame for such properties, describe how sharp the division between tame and wild classes is, and describe the implications or equivalences between these properties. My primary tools are model-theoretic properties of the class and combinatorial properties of infinite structures in the class.

2.1. Completed work.

Theorem 1.2 is an example of this for the property of slow growth rate, with model-theoretic properties used to identify and prove the various gaps and the sharp dichotomy between subexponential growth and growth at least ϕ^n. Behind the growth rate result is the fact that model-theoretic tools also give a detailed recursive characterization of the homogeneous structures with subexponential growth, from which it is also easy to see that the corresponding hereditary classes are well-quasi ordered. Conjecture 1 would be a very strong statement continuing this same line.

For another complexity measure of a hereditary class $\mathcal{C}$, let Sibling($\mathcal{C}$) be the largest cardinal κ such that there is a countable $M \in \mathcal{C}$ and κ many structures (up to isomorphism) are bi-embeddable with M. When $\mathcal{C}$ is a simple class, κ should be small, since bi-embeddability will (nearly) imply isomorphism. Together with Laskowski, we proved the following, confirming some conjectures from [21].

Theorem 2.1 ([9], B., Laskowski). *Suppose $\mathcal{C}$ is a hereditary class in a finite relational language, and let Sibling($\mathcal{C}$) be as in the paragraph above. Then one of the following holds.*

(1) $\mathcal{C}$ is finitely partitioned and Sibling$(\mathcal{C}) = 1$.
(2) $\mathcal{C}$ is cellular and Sibling$(\mathcal{C}) = \aleph_0$.
(3) $\mathcal{C}$ is non-cellular and Sibling$(\mathcal{C}) = 2^{\aleph_0}$.

Again, we see sharp divisions in the complexity as measured by Sibling($\mathcal{C}$), and that these divisions are proved by finding corresponding model theoretic properties. Further, as cellularity also appeared in Theorem 1.2, the complexity of Sibling($\mathcal{C}$) partially aligns with the growth rate of $\mathcal{C}$.

Although it does not appear in the statement of Theorem 2.1, the key model-theoretic dividing line in the proof is *mutual algebraicity*, and the process of proving Theorem 2.1 lead to a finer understanding it. This yielded the companion paper [8], which developed improved quantifier elimination mutually algebraic theories and decompositions for their models, and, for example, proves the following theorem. Together, [8, 9] provide a general approach for showing cellularity is a dividing line for various combinatorial problems.

Theorem 2.2 ([8], B., Laskowski). *M is cellular iff M is mutually algebraic and ω-categorical.*

In the other direction, model-theoretic work with Laskowski about the relation between mutual algebraicity and monadic stability, which lead to [11, 12], was a key inspiration for Theorem 1.2.

Work towards Conjecture 1 has lead to a renewal of interest in the model-theoretic property of *monadic NIP* (condition (2) in the conjecture). With Laskowski, we proved several characterizations of monadic NIP, including the behavior of an independence relation, a forbidden configuration, models admitting certain decompositions into simpler parts, and the behavior of indiscernibles.

Theorem 2.3 ([10] B., Laskowski). *The following are equivalent for a complete theory T.*

(1) T is monadically NIP.
(2) (Let $a \mathop{\smile\!\!\!\!\!\!\,|}^{fs}_M b$ means $\mathrm{tp}(a/Mb)$ is finitely satisfiable in M.) If $a \mathop{\smile\!\!\!\!\!\!\,|}^{fs}_M b$, then for any c, either $ac \mathop{\smile\!\!\!\!\!\!\,|}^{fs}_M b$ or $a \mathop{\smile\!\!\!\!\!\!\,|}^{fs}_M bc$.
(3) (Roughly) T forbids a definable infinite grid.
(4) (Roughly) Models of T can be decomposed into a linearly ordered collection of sets such that each set is finitely satisfiable over its predecessors in a small model. Thus the sets relate simply to each other.
(5) T is dp-minimal and has indiscernible-triviality (if $\mathcal{I}$ is indiscernible over a and over b, then $\mathcal{I}$ is indiscernible over ab).

Corollary 2.4 ([10] B., Laskowski)**.** *Let M be a countable homogeneous structure that is not monadically NIP, $\mathcal{C}$ be the hereditary class of unlabeled substructures of M, and $\mathcal{C}_n$ the elements of $\mathcal{C}$ of size n.*

(1) $|\mathcal{C}_n| = \Omega((n/k)!)$ for some k. (A strengthening of "(1) $\Rightarrow$ (2)" from Conjecture 1.)
(2) $\mathcal{C}$ is not 4-well-quasi-ordered. (A weakening of "(3) $\Rightarrow$ (2)" from Conjecture 1.)

2.2. **Future work.** Conjecture 1, together with results on nowhere-denseness and bounded twin-width, which agree with monadic NIP in monotone and ordered graph classes respectively, indicate that monadic NIP is a fundamental dividing line for hereditary classes, and so a further understanding of monadic NIP is the most promising direction for future work. Additionally, understanding monadic NIP theories, and in particular classifying the homogeneous monadically NIP theories, is a natural next step towards developing a classification theory for general NIP theories.

Conjecture 1 is also closely connected to an old conjecture of Pouzet, concerning when a class $\mathcal{C}$ is *n-well-quasi-ordered*, i.e. every expansion of $\mathcal{C}$ by $n-1$ unary predicates is well-quasi-ordered. A first connection is that if $\mathcal{C}$ is n-well-quasi-ordered for $n \geq 2$, then it must consist of the finite substructures of an ω-categorical structure (i.e. a reduct of a homogeneous structure).

Conjecture 2. *([26], Pouzet) If $\mathcal{C}$ is 2-well-quasi-ordered, then it is n-well-quasi-ordered for all n.*

We refine this as follows: If $\mathcal{C}$ consists of the finite substructures of an ω-categorical monadically NIP structure, then it is n-well-quasi-ordered for all n. Otherwise, it is not 2-well-quasi-ordered.

Corollary 2.4 proves the second part of the refinement with 5 in place of 2. Strengthening that to prove one direction of the refinement would rely on more careful construction and manipulation of the forbidden grid characterizing monadic NIP from Theorem 2.3, and seems well within reach.

Another close connection is to nowhere-denseness, which has been an extremely powerful notion of sparsity in monotone graph classes, i.e. those forbidding (non-induced) subgraphs. In such classes, nowhere-denseness is equivalent to both monadic stability and monadic NIP. A significant direction in graph theory is trying to generalize nowhere-denseness to general hereditary graph classes, and these model-theoretic properties are the most promising candidates.

Question 1. *Which properties of nowhere-denseness generalize to monadic stability? to monadic NIP?*

How much can understanding monadically stable classes be reduced to understanding nowhere-dense classes?

Finally, although Theorem 1.2 provides an understanding of slow growth rates for classes arising from homogeneous structures, unlabeled growth rates for general hereditary classes are not well understood. The case of polynomial growth for graphs [2] and permutations [19] have been considered, but new phenomena arise at higher arities. The following is a test question for understanding classes with polynomial growth.

Conjecture 3 ([27], Pouzet)**.** *Let $\mathcal{C}$ be a hereditary class of structures in a finite relational language, and $\mathcal{C}_n \subset \mathcal{C}$ the structures of size n. If the growth rate of $\mathcal{C}$ is bounded above by a polynomial, then the generating function of $(|\mathcal{C}_n|)_{n \in \omega}$ is rational.*

It is known that such $\mathcal{C}$ must arise as the substructures of an ω-categorical structure with growth rate below 2^n [27], which thus is a relatively simple monadically NIP structure, giving a path to the conjecture.

3. Homogeneous structures

3.1. Completed work.

3.1.1. *Classification.* Although several properties of homogeneous structures and their automorphism groups are studied, few classes of structures have had their homogeneous structures classified. Since interesting or unusual homogeneous structures often only appear in the course of classification, this is a crucial source of examples for other research programs making use of homogeneous structures.

Answering a question of Peter Cameron [14], I undertook the classification of homogeneous structures in a language with n linear orders. Cameron had completed the case $n = 2$, which can be identified with permutations. I completed the case $n = 3$ by direct combinatorial arguments [6] and conjectured a classification for general n. Structures in the conjectured list were built by adding orders to Λ-*ultrametric spaces*, a generalization of ultrametric spaces allowing the distance to take values in a distributive lattice Λ.

With Pierre Simon we confirmed the conjectured classification for all n. Although such classifications typically rely on extensive amalgamation arguments, as in the $n = 3$ case, our proof instead primarily relied on model-theoretic results constraining linear orders in homogeneous structures [28].

Theorem 3.1 ([13] B., Simon). *The homogeneous finite-dimensional permutation structures are as conjectured. In particular, they are interdefinable with expansions of Λ-ultrametric spaces by certain linear orders.*

3.1.2. *Ramsey theory.* Ramsey's theorem is widely used in model theory and combinatorics to extract a well-behaved substructure from a large, complicated structure. Nešetřil and Rödl began the study of Ramsey classes, significantly generalizing Ramsey's theorem.

Definition 3.2. Let $\mathcal{K}$ be a class of structures. Given $A, B \in \mathcal{K}$, let $\binom{B}{A}$ be the set of substructures of B that are isomorphic to A. We say $\mathcal{K}$ is a *Ramsey class* if for any $n \in \mathbb{N}$ and $A, B \in \mathcal{K}$, there is a $C \in \mathcal{K}$ such that if $\binom{C}{A}$ is n-colored, there is a $\widehat{B} \in \binom{C}{B}$ such that $\binom{\widehat{B}}{A}$ is monochromatic.

Under a mild condition, a Ramsey class must consist of the substructures of a homogeneous structure. One of the main questions of the subject is whether every finitely relational homogeneous structure can be expanded by adding finitely many relations so its substructures form a Ramsey class.

The main theorem of [5] identifies the minimal Ramsey expansion for the Λ-ultrametric spaces from Theorem 3.1, i.e. the relations that must be added to make the class of substructures Ramsey, and provides a restatement in terms of topological dynamics. This was an interesting class of structures because finding the Ramsey expansion required using a recent framework of Hubička and Nešetřil [17], the theorem now provides a blueprint for finding Ramsey expansions for other classes with non-refining definable equivalence relations, and it is one of few classes with non-unary algebraic closure whose Ramsey expansion has been found.

Theorem 3.3 ([5], B.). *Let Λ be a finite distributive lattice and Γ be the generic homogeneous Λ-ultrametric space. For every meet-irreducible $E \in \Lambda$, expand Γ by a generic subquotient order from E to its successor, let $\vec{\Gamma}^{min} = (\Gamma, (<_{E_i})_{i=1}^n)$ be structure thus obtained, and $\vec{\mathcal{A}}_\Lambda^{\min}$ its finite substructures.*

(1) $\vec{\mathcal{A}}_\Lambda^{\min}$ is a Ramsey class and has the expansion property relative to $\mathcal{A}_\Lambda$.

(2) The logic action of $Aut(\Gamma)$ on $\overline{Aut(\Gamma) \cdot (<_{E_i})_{i=1}^n}$ is the universal minimal flow of $Aut(\Gamma)$.

Furthermore, every homogeneous structure from Theorem 3.1 is a Ramsey expansion of Γ.

3.2. **Future work.** They Λ-ultrametric spaces from Theorem 3.1 served as one of the motivating for ongoing work by Hubička, Konečný, and Nešetřil on semigroup-valued metric spaces (e.g. [18]). These promise to give a unified description of a wide class of homogeneous structures.

Conjecture 4 ([18], Konečný). *Let M be a primitive homogeneous structure with strong amalgamation in a finite binary symmetric language. Then M can be viewed as a generic semigroup-valued metric space with Henson constraints.*

Classifications, and the standard examples of homogeneous structures, are almost exclusively in binary languages. Higher arities may provide interesting examples for a wide range of questions. For example, one homogeneous 3-hypertournament from [15] provably cannot have its Ramsey expansion identified by the general framework of [17] that suffices for nearly all known cases.

Problem 1. *Find interesting examples of homogeneous structures in a non-binary language. In particular, classify the homogeneous 3-hypertournaments with forbidden substructures of size at most 5.*

We now turn to an infinitary Ramsey property, closely related to Ramsey classes.

Definition 3.4. A structure M has *finite big Ramsey degrees* if for every substructure A, there is a number d_A such that for any coloring of $\binom{M}{A}$ with finitely many colors, there is $N \subset M$ such that $N \cong M$ and $\binom{N}{A}$ has at most d_A colors.

A structure M is generally shown to have finite big Ramsey degrees by representing it as an infinite tree and then proving and applying a suitable Ramsey theorem for such trees. The program of showing various structures have finite big Ramsey degrees in turn provides interesting examples for set-theoretic combinatorics, such as the theory of Ramsey spaces.

Currently, relatively few structures are known to have finite big Ramsey degrees. But this may mirror the early status of Ramsey classes, which were once thought to be rare, but general tools were developed to identify them and they are now conjectured to be ubiquitous.

Question 2. *Do all homogeneous structures in a finite relational language with strong amalgamation have finite big Ramsey degrees?*

Finally, the process of representing a homogeneous structure as a tree and computing its big Ramsey degrees in some cases yields its minimal Ramsey expansion, in the sense of the previous subsection.

Problem 2. *Clarify the connection between finding the minimal Ramsey expansion for a homogeneous structure and computing its big Ramsey degrees.*

References

[1] Peter Allen, Vadim Lozin, and Michaël Rao, *Clique-width and the speed of hereditary properties*, The Electronic Journal of Combinatorics (2009), R35.

[2] József Balogh, Béla Bollobás, Michael Saks, and Vera T Sós, *The unlabelled speed of a hereditary graph property*, Journal of Combinatorial Theory, Series B **99** (2009), no. 1, 9–19.

[3] Achim Blumensath, *A model-theoretic characterisation of clique width*, Annals of Pure and Applied Logic **142** (2006), no. 1-3, 321–350.

[4] Édouard Bonnet, Ugo Giocanti, Patrice Ossona de Mendez, and Stéphan Thomassé, *Twin-width IV: low complexity matrices*, arXiv preprint arXiv:2102.03117 (2021).

[5] Samuel Braunfeld, *Ramsey expansions of Λ-ultrametric spaces*, arXiv preprint arXiv:1710.01193 (2017).

[6] ———, *Homogeneous 3-Dimensional Permutation Structures*, arXiv preprint arXiv:1710.05138 (2017).

[7] ———, *Monadic stability and growth rates of ω-categorical structures*, arXiv preprint arXiv:1910.04380 (2019).

[8] Samuel Braunfeld and Michael C Laskowski, *Mutual algebraicity and cellularity*, arXiv preprint arXiv:1911.06303 (2019).

[9] ———, *Counting siblings in universal theories*, arXiv preprint arXiv:1910.11230 (2019).

[10] ———, *Characterizations of monadic NIP*, arXiv preprint arXiv:2104.12989 (2021).

[11] ———, *Worst case expansions of complete theories*, arXiv preprint arXiv:2107.10920 (2021).

[12] ———, *Theories admitting congruences over sets and boundedness*, arXiv preprint arXiv:2109.08943 (2021).

[13] Samuel Braunfeld and Pierre Simon, *The classification of homogeneous finite-dimensional permutation structures*, The Electronic Journal of Combinatorics **27** (2020), no. 1, Research Paper 38.

[14] Peter J Cameron, *Homogeneous permutations*, The Electronic Journal of Combinatorics **9** (2002), no. 2, Research Paper 2.

[15] Gregory Cherlin, Jan Hubička, Matěj Konečný, and Jaroslav Nešetřil, *Ramsey expansions of 3-hypertournaments*, arXiv preprint arXiv:2105.12368 (2021).

[16] Zdeněk Dvořák and Serguei Norine, *Small graph classes and bounded expansion*, Journal of Combinatorial Theory, Series B **100** (2010), no. 2, 171–175.

[17] Jan Hubička and Jaroslav Nešetřil, *All those Ramsey classes*, arXiv preprint arXiv:1606.07979 (2016).

[18] Matěj Konečný, *Semigroup-valued metric spaces*, arXiv preprint arXiv:1810.08963 (2018).

[19] Tomáš Kaiser and Martin Klazar, *On growth rates of closed permutation classes*, The Electronic Journal of Combinatorics (2002), R10.

[20] Alexander S Kechris, Vladimir G Pestov, and Stevo Todorčević, *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups*, Geometric and Functional Analysis **15** (2005), no. 1, 106–189.

[21] Claude Laflamme, Maurice Pouzet, Norbert Sauer, and Robert Woodrow, *Siblings of an $\aleph_0$-categorical relational structure*, arXiv preprint arXiv:1811.04185 (2018).

[22] Vadim Lozin, Igor Razgon, and Viktor Zamaraev, *Well-quasi-ordering versus clique-width*, Journal of Combinatorial Theory, Series B **130** (2018), 1–18.

[23] HD Macpherson, *Orbits of infinite permutation groups*, Proceedings of the London Mathematical Society **3** (1985), no. 2, 246–284.

[24] ———, *Growth rates in infinite graphs and permutation groups*, Proceedings of the London Mathematical Society **3** (1985), no. 2, 285–294.

[25] Jaroslav Nešetřil and Patrice Ossona de Mendez, *Sparsity: graphs, structures, and algorithms*, Vol. 28, Springer Science & Business Media, 2012.

[26] Maurice Pouzet, *The profile of relations*, arXiv preprint math/0703211 (2007).

[27] ———, *Un bel ordre d'abritement et ses rapports avec les bornes d'une multirelation*, CR Acad. Sci. Paris Sér. AB **274** (1972), A1677–A1680.

[28] Pierre Simon, *NIP omega-categorical structures: the rank 1 case*, arXiv preprint arXiv:1807.07102 (2018).

[29] Pierre Simon and Szymon Toruńczyk, *Ordered graphs of bounded twin-width*, arXiv preprint arXiv:2102.06881 (2021).