RESEARCH STATEMENT

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1. Introduction

My research lies in the intersection of model theory and combinatorics. Model theory provides several dividing lines separating tame from wild behavior in certain classes of (infinite) structures. Much of my research has been an interplay between purely model-theoretic results about these dividing lines and their application to the combinatorics of hereditary classes of (finite) structures, i.e. classes closed under taking substructures. These applications include both solutions to concrete problems such as classification and enumeration, and results broadly describing the "geography" of such classes (see Figure 1 for a partial map).

The sparsity program of Nešetřil and Ossona de Mendez, with its central nowhere dense/somewhere dense dividing line for the more restricted setting of monotone graph classes, has yielded innumerable structural and algorithmic results, and many researchers are actively searching for an analogous dividing line in general hereditary classes, which would also capture dense graphs. A favored approach is to use the observation of [1] that in monotone graph classes, nowhere denseness is equivalent to the classical model-theoretic dividing lines of stability and NIP, as well as the stronger and less-studied variants monadic stability and monadic NIP, and to thus try to understand these properties in hereditary classes.

In my work, joint with Laskowski [6], we have given several characterizations of monadic NIP in first order theories (Theorem 2.3), including the behavior of an independence relation (generalizing e.g. linear independence), the ability to decompose infinite models into a sequence of independent pieces, and a single structure that serves as a canonical obstruction. More recently [7], we have improved these results in the setting of hereditary classes, which for example shows that monadic NIP and NIP agree in hereditary classes, as do stability and monadic stability (Theorem 2.4). This gives an explanation for the appearance in hereditary classes of these monadic variants of classical dividing lines, and simplifies the landscape by showing there are only two model-theoretic generalizations of nowhere denseness to consider, rather than four.

For an application of such results to a concrete problem, I had used earlier analogous characterizations of monadic stability to confirm some conjectures of Peter Cameron and Dugald Macpherson [10, 17, 18] on orbit counting. Generalizing the classical problem of counting orbits of a finite group action, let a group G act on a countable set and define the function $f_G(n)$ counting the number of orbits on n-sets. Many familiar sequences appear as such functions, and their asymptotic behavior is seemingly tightly constrained, exhibiting narrow bands with large gaps between them. In [4], I proved the existence of these gaps and nearly completely classified the possible sub-exponential asymptotics of $f_G(n)$ (Theorem 2.1), by identifying the group G with the automorphism group of a highly symmetric countable structure M and showing the model-theoretic complexity of M controls the asymptotics of $f_G(n)$, with jumps in one giving jumps in the other.

This problem intersects another strand of my research, which is to study the automorphism groups of highly symmetric (i.e. *ω-categorical*) countable structures as above, and how their dynamical or group-theoretic properties relate to the model theory of the structure and to the combinatorics of its finite substructures. For example, a paradigmatic result from [15] shows that for such M, the strong dynamical property of extreme amenability for $Aut(M)$ is equivalent to the finite substructures of M being a Ramsey class (Definition 3.2).

The extra constraint of ω-categoricity makes a fine structural understanding of certain model-theoretic classes feasible, and historically these analyses have given rise to concepts and techniques leading to an understanding of these model-theoretic classes in broader generality. In this line, and ultimately jointly with Pierre Simon, I have provided a full classification of certain ω-categorical structures (Theorem 3.1), showing how the presence of linear orders in NIP theories gives very strong structural information.

This makes the classification of ω-categorical monadically NIP theories a natural target, which should develop tools for understanding the structure of models of (monadically) NIP theories in general. Indeed, my results on monadic NIP mentioned at the beginning were spurred by the following conjecture connecting it to asymptotic enumeration, structural graph theory, algorithmic complexity, and monadic second-order logic.

Conjecture 1. *Let M be an ω-categorical structure and $\mathcal{C}$ be the hereditary class of substructures of M. Then the following are equivalent.*

(1) M is monadically NIP.
(2) $f_{Aut(M)}(n) = O(c^n)$ for some $c \in \mathbb{R}$.
(3) $\mathcal{C}$ is well-quasi-ordered by the substructure relation, i.e. it contains no infinite antichain.
(4) $\mathcal{C}$ has finite clique-width.

(5) First-order model checking in $\mathcal{C}$ is fixed-parameter tractable.
(6) M has a decidable monadic second-order theory.

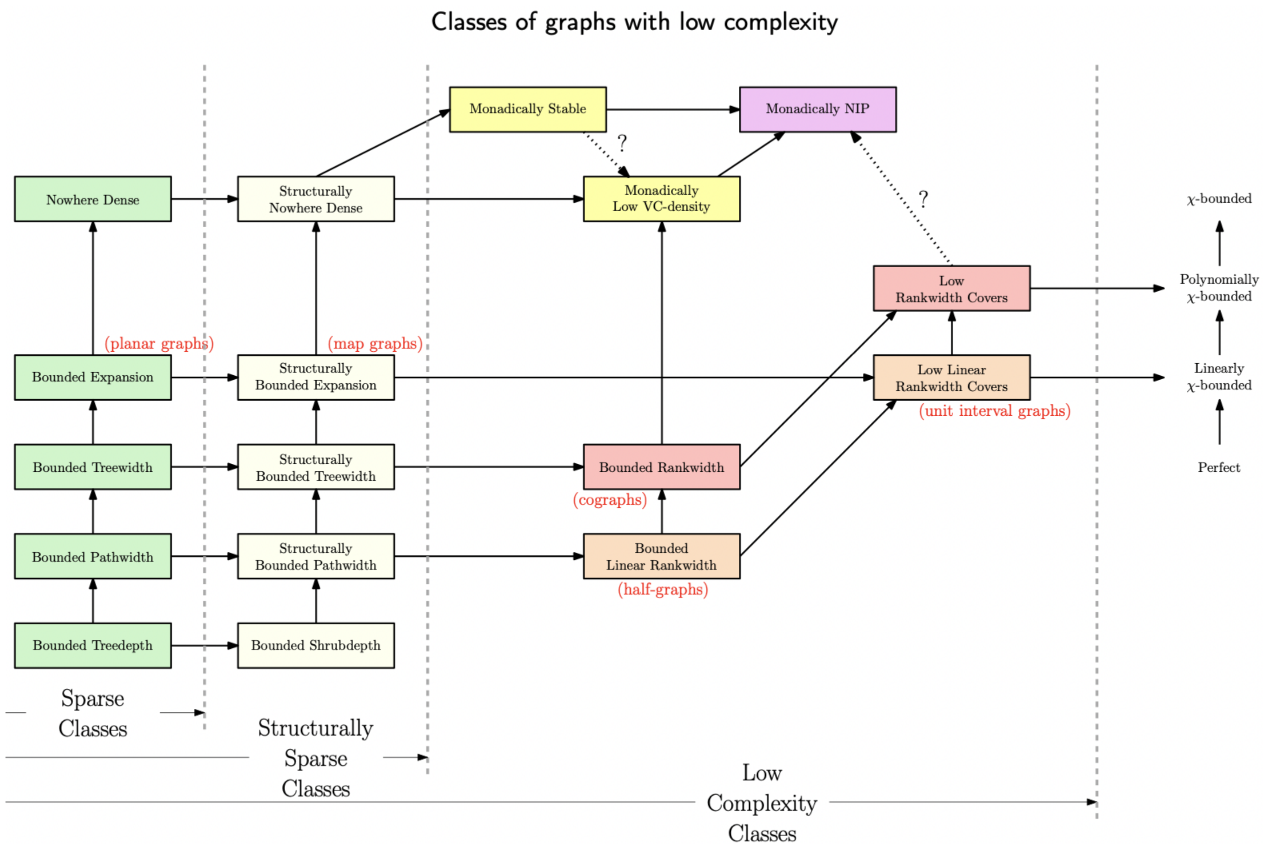

FIGURE 1. ([21]) Inclusions of some properties of graph classes. Example classes are given in parentheses.

2. Model theory and tame hereditary classes

A hereditary class of relational structures, i.e. one closed under taking substructures, admits many measures of combinatorial complexity or tameness: the size of the class (e.g. the asymptotics of the number of structures of size n for each n), the complexity of (bi-)embeddability (e.g. an infinite antichain under embedding), some notion of bounded width (e.g. bounded tree-width), admitting efficient algorithms to generally intractable problems, and so on. Ultimately, we would like a structure theory for the tame classes describing how structures in the class can be decomposed into simple parts, and from which other tameness results follow.

Model theory provides another way to determine the complexity of hereditary classes based on which other hereditary classes they can define, and has benchmark properties for separating the tame classes from the wild. Such properties include NFCP, stability, NIP, and their monadic variants that also require these properties to be preserved under arbitrary expansions by unary predicates (i.e. arbitrary colorings of points). Ideally, some such model-theoretic property corresponds to the divide between tame and wild behavior for a combinatorial complexity property under study, and this can be proved in two steps: If the model-theoretic property fails, then this is witnessed by the presence of a canonical structure that can be manipulated to produce the wild behavior. If the model-theoretic property holds, then this gives additional information, often in the form of the behavior of a suitable independence relation, that can be used to decompose the structures into independent pieces, yielding the desired structure theory.

2.1. **Selected completed work.** We start with two examples of the ideal framework outlined above. The first is the theorem on the orbit-counting function $f_G(n)$ mentioned in the introduction, which can also be viewed as counting the structures of size n in certain hereditary classes. The sharp division between subexponential growth and growth at least ϕ^n corresponds to monadic stability, and the different behaviors in the subexponential growths correspond to the parenthetical properties.

Theorem 2.1 ([4], B.)**.** *Let a group G act on a countable set X. If $f_G(n) = o(\phi^n)$, for ϕ the golden ratio, then $f_G(n) = o(c^n)$ for every $c > 1$. Furthermore, one of the following holds.*

1. *(Cellularity) There are $c > 0$, $k \in \mathbb{N}$ such that $f_G(n) \sim cn^k$.*
2. *(Monadic stability, Morley rank 2) There are $c > 0$, $k \in \mathbb{N}$ such that $f_G(n) = e^{\left(\Theta\left(n^{1-\frac{1}{k}}\right)\right)}$.*
3. *(Monadic stability, Morley rank $r+2$) Let $\log^r(n)$ denote the r-fold iterated logarithm. There are $c > 0$ and k, $r \in \mathbb{N}$ such that $f_G(n) = e^{\left(\Theta\left(\frac{n}{(\log^r(n))^{1/k}}\right)\right)}$.*

For another complexity measure of a hereditary class $\mathcal{C}$, let Sibling($\mathcal{C}$) be the largest cardinal κ such that there is a countable $M \in \mathcal{C}$ and κ many structures (up to isomorphism) are bi-embeddable with M. When $\mathcal{C}$ is a simple class, κ should be small, since bi-embeddability will (nearly) imply isomorphism. With Laskowski, we proved the following, confirming conjectures from [16]. Again, we see sharp divisions in the complexity as measured by Sibling($\mathcal{C}$), and these divisions are proved by finding corresponding model-theoretic properties.

Theorem 2.2 ([5], B., Laskowski)**.** *Suppose $\mathcal{C}$ is a hereditary class in a finite relational language, and let Sibling($\mathcal{C}$) be as in the paragraph above. Then one of the following holds.*

1. *$\mathcal{C}$ is finitely partitioned and Sibling($\mathcal{C}$) $= 1$.*
2. *$\mathcal{C}$ is cellular and Sibling($\mathcal{C}$) $= \aleph_0$.*
3. *$\mathcal{C}$ is non-cellular and Sibling($\mathcal{C}$) $= 2^{\aleph_0}$.*

Such applications need an understanding of the corresponding model-theoretic property and its failure. With Laskowski, we proved several characterizations of monadic NIP, stated vaguely below. The various characterizations all indicate that models of monadically NIP theories are coarsely "linear order-like".

Theorem 2.3 ([6] B., Laskowski)**.** *The following are equivalent for a complete theory T (vaguely stated).*

1. *T is monadically NIP.*
2. *The independence relation induced by finite satisfiability has additional properties.*
3. *Models of T do not encode a particular grid-like structure.*
4. *Models of T can be decomposed into a linear sequence of independent sets.*
5. *A type-counting bound, similar to bounding the linear clique-width of models by a cardinal.*
6. *Indiscernible sequences behave well after naming parameters.*

Recently, we have sharpened Theorem 2.3 in the case of universal theories by partially working in the setting of existentially closed structures. One application is the following theorem, tidying the "model-theoretic map" of hereditary classes. The third part answers questions from [1, 19].

Theorem 2.4. *Let $\mathcal{C}$ be a hereditary class of relational structures.*

1. *$\mathcal{C}$ is stable if and only if $\mathcal{C}$ is monadically stable.*
2. *$\mathcal{C}$ is NIP if and only if $\mathcal{C}$ is monadically NIP.*
3. *Furthermore, if $\mathcal{C}$ is monotone (a generalization of closure under graph-theoretic subgraph), then all four of these properties agree.*

2.2. **Future work.** We mention a few directions in the main goal of further understanding (monadic) stability and NIP in hereditary classes, and how they relate to notions and constructions from structural graph theory.

One direction is to try to characterize stable/NIP graph classes using notions from structural graph theory, such as low covers or forbidden shallow vertex minors. Low covers are roughly a means of considering graph classes that locally have a particular property. They are a powerful tool for reducing problems to simpler classes, and seem to be a fundamental construction. For example, starting from bounded treedepth, n^ϵ-low covers yield nowhere denseness and $O(1)$-low covers yield bounded expansion (see Figure 1), which is the other main dividing line in the sparsity program. The model-theoretic result behind Theorem 2.4 allow us to lift local to global structure and show that low covers preserve stability and NIP (confirming a tentative arrow in Figure 1, and much more) [8], and thus they are natural properties for this construction.

Question 1. *Can stability and NIP be characterized in terms of admitting low covers from better-understood properties? For example, is a hereditary class NIP if and only if it admits n^ϵ-low twin-width covers?*

The standard definition of nowhere dense classes is by forbidding some clique as a shallow minor, a local way of forbidding substructures. The notion of vertex minors seem to be the appropriate analogue of minors in hereditary classes, and there is active work on developing a structure theory for classes forbidding vertex minors in parallel to the Robertson-Seymour theory for forbidden minors [13]. This suggests the following.

Question 2. *Can stability and NIP be characterized in terms of forbidden shallow vertex minors? For example, is a hereditary class NIP if and only if it forbids some circle graph as a shallow vertex minor?*

In addition to low covers, another way to reduce questions about a class $\mathcal{C}$ to a class $\mathcal{D}$ is to show that $\mathcal{C}$ can be defined in a unary expansion of $\mathcal{D}$. The major conjecture here is the following, which would collapse "monadically stable" to "structurally nowhere dense" in Figure 1. For example, the figure shows this would imply the conjecture that monadically stable classes have low VC-density.

Conjecture 2. *Every stable graph class is definable in a unary expansion of a nowhere dense class.*

One approach to this would be to try to finitize the tree decomposition for monadically stable theories, which in its present form is only informative for uncountable models. Since the decomposition is in terms of non-forking independence, which is still well-understood over finite sets, this is feasible, and may connect to known tree-decompositions for nowhere dense classes.

Finally, we turn to analytic limits of classes of finite structures. Given a fragment $X \subseteq FO$ of first order logic and a suitably convergent class, the X-modeling limits of Nešetřil and Ossona de Mendez provide a relational structure on a standard Borel probability space in which all X-definable sets are measurable. For quantifier-free formulas, the convergence notion agrees with the left convergence for graphons, while for local formulas with one free variable it agrees with the local convergence of Benjamini-Schramm. It is known that every nowhere dense graph class admits an FO-modeling limit [20]. But it is open whether they admit strong FO-modeling limits, roughly where the measure is preserved as it flows across relations between subsets. Generalizing the Aldous-Lyons conjecture, this extra property may essentially be sufficient to guarantee a Borel structure is the limit of some FO-convergent class.

Conjecture 3 ([20])**.** *Let $\mathcal{C}$ be a nowhere dense graph class. Then $\mathcal{C}$ admits a strong FO-modeling limit.*

The proof of the existence of FO-modeling limits for nowhere dense classes crucially uses a completeness theorem for Friedman's logic with a quantifier for "there exist positive measure many" [25], proved by carefully constructing a Borel model based on an indiscernible sequence. A better understanding of this construction ought to shed light on the conjecture.

More broadly, since FO-modeling limits are pseudofinite, we can ask the following.

Question 3. *Does the model theory of pseudofinite structures, such as the coarse and fine pseudofinite dimensions or asymptotic classes, have any bearing on FO-modeling limits?*

3. ω-categorical and homogeneous structures

3.1. Completed work.

3.1.1. *Classification.* The connections between properties of ω-categorical structures, of their automorphism groups, and of their finite substructures are intensively studied, relating model theory, dynamics, and combinatorics. The classification of homogeneous structures in a fixed language, which are defined by the property of having all finite partial automorphisms extend to full automorphisms, is a key source of examples for this field, since a classification will lead to any exotic structures of the given type that may exist.

Answering a question of Peter Cameron [11], I undertook the classification of homogeneous structures in a language with n linear orders. After constructing conjectural list, with structures built by adding orders in definite ways to Λ-*ultrametric spaces*, a generalization of ultrametric spaces allowing the distance to take values in a distributive lattice Λ, I confirmed the conjectured classification for all n in joint work with Pierre Simon. Although such classifications typically rely on extensive amalgamation arguments, our proof instead primarily relied on model-theoretic results constraining linear orders in ω-categorical structures [24].

Theorem 3.1 ([9] B., Simon)**.** *The homogeneous structures in a language of finitely many linear orders are as conjectured, i.e. they are interdefinable with expansions of Λ-ultrametric spaces by certain linear orders.*

This was the first step extending the analysis of [24] beyond the setting of thorn-rank 1. The finite rank setting was later addressed in [23]. The case of finitely homogeneous monadically NIP structures as considered in Conjecture 1 is the natural next step beyond finite rank, allowing for tree-like structures. In another outcome from Theorem 3.1, Λ-ultrametric spaces were a motivating example for the definition of semigroup-valued metric spaces [14]. These are conjectured to account for all homogeneous structures in a language with symmetric binary relations, providing a unified view of seemingly diverse structures.

3.1.2. *Ramsey theory.* Ramsey's theorem is widely used in model theory and combinatorics to extract a well-behaved substructure from a large, complicated structure. Nešetřil and Rödl began the study of Ramsey classes [22], significantly generalizing Ramsey's theorem.

Definition 3.2. Let $\mathcal{K}$ be a class of structures. Given $A, B \in \mathcal{K}$, let $\binom{B}{A}$ be the set of substructures of B that are isomorphic to A. We say $\mathcal{K}$ is a *Ramsey class* if for any $n \in \mathbb{N}$ and $A, B \in \mathcal{K}$, there is a $C \in \mathcal{K}$ such that if $\binom{C}{A}$ is n-colored, there is a $\widehat{B} \in \binom{C}{B}$ such that $\binom{\widehat{B}}{A}$ is monochromatic.

Under a mild condition, a Ramsey class must consist of the substructures of a homogeneous structure. One of the main questions of the subject is whether every finitely relational homogeneous structure can be expanded by adding finitely many relations so its substructures form a Ramsey class.

In [3], I identified the minimal Ramsey expansion for the Λ-ultrametric spaces from Theorem 3.1, i.e. the relations that must be added to make the class of substructures Ramsey, and provided a restatement in terms of topological dynamics. The theorem now provides a blueprint for finding Ramsey expansions for other classes with non-refining definable equivalence relations, and gives one of few classes with non-unary algebraic closure whose Ramsey expansion has been found.

Theorem 3.3 ([3], B.)**.** *Let Λ be a finite distributive lattice and Γ be the generic homogeneous Λ-ultrametric space, with finite substructures $\mathcal{A}_\Lambda$. The minimum Ramsey expansion of $\mathcal{A}_\Lambda$ is explicitly given, which is equivalent to determining the universal minimal flow of $Aut(\Gamma)$. Furthermore, every homogeneous structure from Theorem 3.1 is a Ramsey expansion of such a Γ.*

3.2. **Future work.** Again, Conjecture 1 and the classification of ω-categorical monadically NIP structures are major questions and relate this theme of my research with the previous section. Rather than discuss it further, we continue to the topic of finite big Ramsey degrees. Here one takes an infinite structure M, colors the copies of a finite substructure A, and searches for a monochromatic copy of M (or more generally, a copy with a bounded number of colors).

A structure M is generally shown to have finite big Ramsey degrees by representing it as an infinite tree and then proving and applying a suitable Ramsey theorem for such trees, so the techniques are primarily infinitary/set-theoretic combinatorics. Currently, relatively few structures are known to have finite big Ramsey degrees. But this may mirror the early status of Ramsey classes, which were once thought to be rare, but general tools were developed to identify them and they are now conjectured to be ubiquitous.

Question 4. *Is there a homogeneous structure in a finite relational language with strong amalgamation and finitely many forbidden substructures that does not have finite big Ramsey degrees?*

Exact big Ramsey degrees for binary free amalgamation classes were recently characterized [2], but even the finiteness problem for higher arities is significantly more complicated, and the subject of ongoing joint work with several collaborators.

Problem 1. *Does every homogeneous structure in a finite relational language with free amalgamation and finitely many forbidden substructures have finite big Ramsey degrees?*

We close by mentioning automorphism-invariant measures on definable sets of ω-categorical structures. These generalize the classical model-theoretic notion of invariant types, and the problems connect to higher-order generalizations of Szemerédi regularity, and the associated notions of graphons from graph limit theory and exchangeable arrays from probability. In ongoing work with Paolo Marimon, we have disproved a conjecture [12] that all ergodic measures on the countable random 3-uniform hypergraph are Bernoulli, showing that sampling from any graphon yields such a measure. These measures are "degenerate" in the sense that they depend only on binary data, reminiscent of results in higher-order Szemerédi regularity.

Conjecture 4. *Every ergodic automorphism-invariant measure on the definable sets of the countable random 3-uniform hypergraph is produced as above by sampling from a graphon.*

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