Name and Surname:
Pseudonym:

1	2	3	4	5	6

5A. Exam Paper NMAI059 Probability and Statistics 1 – 24.6.2024

On each paper, write the problem number and your surname.

You can also write your chosen pseudonym on this paper, under which your results will be published. (Otherwise, they will be published with your initials.) **Also submit the problem sheet (it will be available on the web).**

Do not write multiple problems on the same page! You have **150 minutes** to complete the work.

No calculators, mobile phones, etc. are allowed during the work. (Please turn off mobile phones in advance.) **During the entire exam, mobile phones must be stored in a closed bag.**

If the result contains expressions that are difficult to calculate without a calculator, do not evaluate them: $137 \cdot 173$ is as good, if not a better answer, than 23701.

Justify all calculations in detail! Even a correct result without justification is almost worthless.

You can use one (handwritten) A4 formula sheet.

After correcting the exam, a grade of 1, ..., 5 will be proposed to everyone via email. You can improve this by one grade during the oral part – i.e., 4 can be improved to 3, but 5 means failure in this exam term.

You might find the following tables useful.

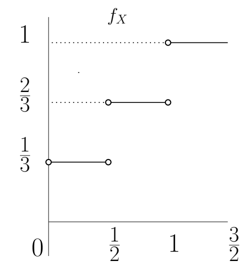
x	-2.0	-1.5	-1.0	-0.5	0.0	0.5	1.0	1.5	2.0
$\Phi(x)$	0.02	0.07	0.16	0.31	0.5	0.69	0.84	0.93	0.98
$\Psi_1(x)$	0.15	0.19	0.25	0.35	0.5	0.65	0.75	0.81	0.85
$\Psi_2(x)$	0.09	0.14	0.21	0.33	0.5	0.67	0.79	0.86	0.91
$\Psi_9(x)$	0.04	0.08	0.17	0.31	0.5	0.69	0.83	0.92	0.96

p	0.9	0.95	0.975	0.99	0.995
$\Phi^{-1}(p)$	1.2816	1.6449	1.96	2.3263	2.5758
$\Psi_1^{-1}(p)$	3.0777	6.3138	12.7062	31.8205	63.6567
$\Psi_2^{-1}(p)$	1.8856	2.92	4.3027	6.9646	9.9248
$\Psi_{59}^{-1}(p)$	1.2961	1.6711	2.001	2.3912	2.6618
$\Psi_{60}^{-1}(p)$	1.2958	1.6706	2.0003	2.3901	2.6603
$\Psi_{61}^{-1}(p)$	1.2956	1.6702	1.9996	2.389	2.6589

Justify all calculations in detail! Even a correct result without justification is almost worthless.

1. (10 points) The figure shows the density of a random variable X. Outside the marked interval, this function is zero.

(a) Verify that it is a density function.
(b) Is it a discrete or continuous random variable?
(c) Calculate $P(X < 1)$.
(d) What is $P(X \geq 1/2)$?
(e) What is the median of X?
(f) What is the eightieth percentile of X?
(g) Calculate $\mathbb{E}(X)$.



2. (10 points) During a storm, we will observe the number of lightning flashes. For simplicity, assume that it never flashes twice in one second. For each second, there is a probability $p = 1/100$ of a flash, and the individual seconds are independent.

(a) What is the probability that we will see at least one flash in the next two minutes?
(b) What is the prob. that we will see exactly ten flashes in the next twenty minutes?
(c) Approximate the answer to the previous question using the Poisson distribution.

3. (10 points)

A courier delivers packages around Prague (he delivers them by bike, so he is not affected by traffic and the individual deliveries can be considered independent). The delivery time for the i-th package will be T_i. According to experience, $\mathbb{E}(T_i) = 20$ minutes and $\sigma_{T_i} = 6$ minutes.

(a) What is the mean and standard deviation of the time T for delivering 25 packages?
(b) Use the Central Limit Theorem to estimate the probability that the courier will make it in less than 8 hours.

Write the exact formula using the function Φ and use the table on the previous page for an estimate. Also, state what assumptions we make about the random variables T_i. Under what assumption on T_i does the given probability formula hold exactly?

4. (10 points) (a) Define the concept of covariance of random variables.

Let X be the result of rolling a six-sided die and M the remainder when X is divided by three (i.e., M takes the values 0, 1, 2). Finally, let $Y = M + 2/3$. Determine the covariance of X and Y. Are the variables X and Y independent?

(b) Define the concept of joint cdf. Let $F = F_{X,Y}$ be the joint cdf of the random variables X, Y, given by the formula $F(x, y) = 1 - e^{-x} - e^{-2y} + e^{-x-2y}$ (for $x, y \geq 0$). What is the probability that $0 < X < 1$ and $1 < Y < 2$?

5. (10 points) State the Central Limit Theorem. Explain the ways it is used.

Among other things, use this theorem to estimate the probability that out of 180 rolls of a die, we will get at least 35 sixes.

6. (10 points)

State the Weak Law of Large Numbers. Prove it.