

Name:

1	2	3	4	5	6

1st exam NMAI073 Probability and Statistics 2 – Jan 19, 2024

Write your last name on each sheet of paper.

Do not write more than one problem on the same page. Write the number of the problem clearly on top of the page.

On this paper, you can also write a chosen pseudonym under which your results will be announced. (Otherwise, they will be announced with your initials.) Also submit this paper (it will be available on the website).

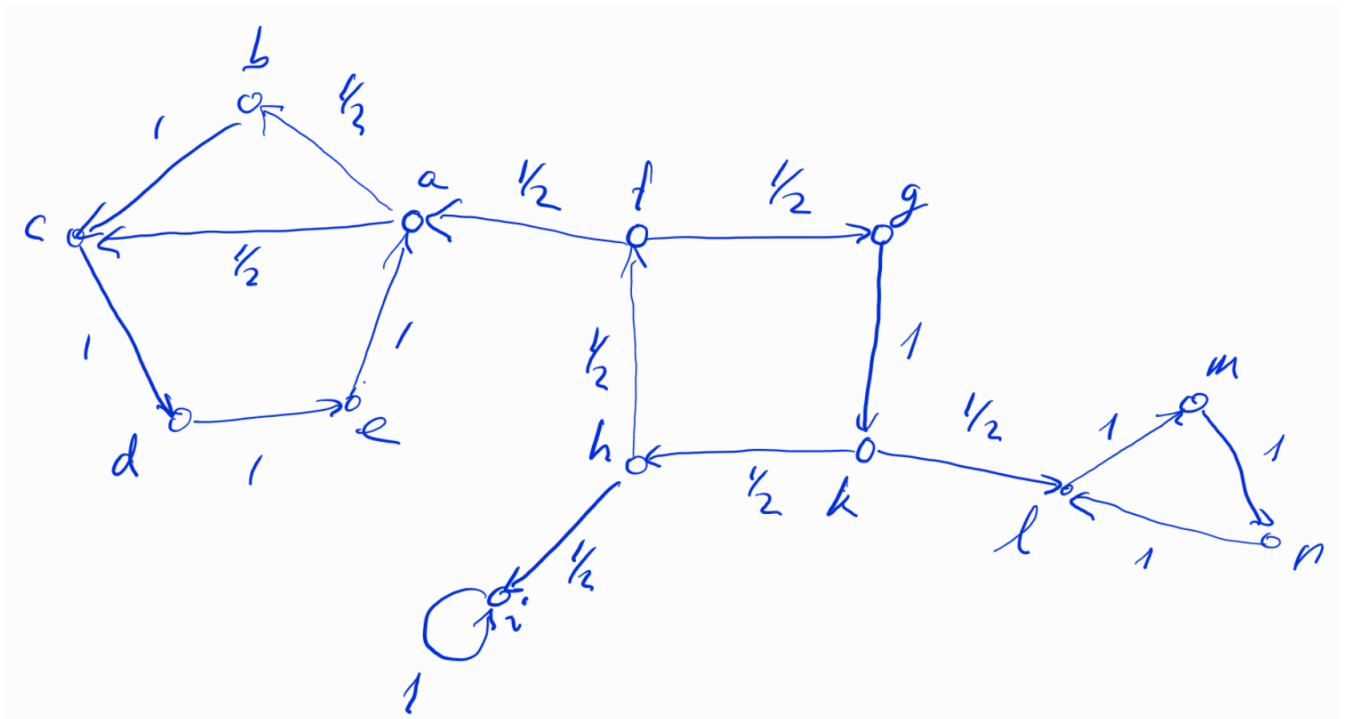
You have **150 minutes** to complete the assignment.

No calculators, mobile phones, etc., are allowed during the work. (Please silence your mobile phones beforehand.)

If the result contains expressions that are difficult to calculate without a calculator, do not compute them: $137 \cdot 173$ is just as good, if not better, an answer than 23701.

Thoroughly justify all calculations.

You may use one (handwritten) cheat sheet of A4 format.



1. (10 points) Consider the Markov chain on the figure.
- (a) Find classes of equivalence \leftrightarrow . You may mark them directly on the figure.
 - (b) Determine which states are transient, recurrent, periodic (and what is the period).
 - (c) Does the distribution converge to a stationary distribution, if we start at state a ?

If we start at f ?

2. (10 points) Let (X, Y) have a joint pdf that is uniform on a triangle with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$.
- (a) Find the joint pdf.
 - (b) Find marginal PDF of Y .
 - (c) Find $f_{X|Y}$.
 - (d) Find $\mathbb{E}(X | Y = y)$. Use total expectation law to find $\mathbb{E}(X)$ in terms of $\mathbb{E}(Y)$.
 - (e) Find $\mathbb{E}(X)$, $\mathbb{E}(Y)$.
 - (f) Find $\mathbb{E}(X | Y)$.

3. (10 points) In Brazilian jungle, biologists are studying a rare species of bird. They try to estimate the proportion of male birds in this species. Prof. A models this proportion by a random variable $A \sim U(0, 1)$, while Prof. B by a random variable with a pdf $f_B(p) = cp$ (for all $p \in (0, 1)$ and suitable c , $f_B(p) = 0$ outside of interval $[0, 1]$).

The biologists capture randomly 5 birds, out of which 3 turn out to be male.

- (a) What is the posterior pdf for the proportion of male birds according to Prof. A? What is the MAP estimate?
- (b) How does the answer differ for Prof. B?
- (c) What other estimate we can use? (You don't need to compute any integral, only express your answer using some integrals.)

4. (10 points) A small café receives customers according to a Poisson process with a rate of $\lambda = 10$ customers per hour. Each customer orders a cup of coffee with a probability of $p = 1/2$, independent of other customers.

- (a) Find the probability that exactly two customers order coffee between 9 and 10 a.m.
- (b) Find the probability that exactly three customers enter café but only two of them order coffee between 10 a.m. and noon.
- (c) Determine the expected time from noon until the first coffee order.
- (d) Calculate the probability that at least one customer orders coffee within the first 30 minutes of the new day.

5. (10 points) Prove central limit theorem using Moment generating functions. (You don't have to prove the required properties of MGF, only state what you are using.)

6. (10 points) State and prove formula for probability of absorption in a Markov chain.