Exercise session 9 – Prob. & Stat. 2 — Nov 28, 2023

The Bernoulli process

1. Each of n packages is loaded independently onto either a red truck (with probability p) or onto a green truck (with probability $1-p$). Let R be the total number of items selected for the red truck and let G be the total number of items selected for the green truck.

(a) Determine the PMF[1], expected value, and variance of the random variable R.

(b) Evaluate the probability that the first item to be loaded ends up being the only one on its truck.

(c) Evaluate the probability that at least one truck ends up with a total of exactly one package.

(d) Evaluate the expected value and the variance of the difference $R-G$.

(e) Assume that $n \geq 2$. Given that both of the first two packages to be loaded go onto the red truck, find the conditional expectation, variance, and PMF of the random variable R.

2. A computer system carries out tasks submitted by two users. Time is divided into slots. A slot can be idle, with probability $p_I = 1/6$, and busy with probability $p_B = 5/6$. During a busy slot, there is probability $p_{1|B} = 2/5$ (respectively, $p_{2|B} = 3/5$) that a task from user 1 (respectively, 2) is executed. We assume that events related to different slots are independent.

(a) Find the probability that a task from user 1 is executed for the first time during the 4th slot.

(b) Given that exactly 5 out of the first 10 slots were idle, find the probability that the 6th idle slot is slot 12.

(c) Find the expected number of slots up to and including the 5th task from user 1.

(d) Find the expected number of busy slots up to and including the 5th task from user 1.

(e) Find the PMF, mean, and variance of the number of tasks from user 2 until the time of the 5th task from user 1.

3. (Sum of a geometric number of independent geometric random variables.)
Let $Y = X_1 + \cdots + X_N$, where the random variables X_i are geometric with parameter p and N is geometric with parameter q. Assume that the random variables $N, X_1, X_2, \ldots$ are independent. Show that Y is geometric with parameter pq. Hint: Interpret the various random variables in terms of a split Bernoulli process.

4. * (The bits in a uniform random variable form a Bernoulli process.)
Let $X_1, X_2, \ldots$ be a sequence of binary random variables taking values in the set $\{0,1\}$. Let Y be a continuous random variable that takes values in the set $[0,1]$. We relate X and Y by assuming that Y is the real number whose binary representation is $0.X_1X_2X_3\ldots$

More concretely

$$Y = \sum_{i \geq 1} 2^{-i} X_i.$$

(a) Suppose that the X_i form a Bernoulli process with parameter $p = 1/2$. Show that Y is uniformly distributed. [Hint: Consider the probability of the event $(i-1)/2^k < Y < i/2^k$, where i and k are positive integers.]

(b) Suppose that Y is uniformly distributed. Show that the $X_1, X_2, \ldots$ form a Bernoulli process with parameter $p = 1/2$.

[1] probability mass function, pravděpodobnostní funkce