Exercise session 2 – Prob. & Stat. 2 — Oct 10, 2023

Classification of states

A state i has *period* $d_i := \gcd\{t : r_{i,i}(t) > 0\}$. It is called *periodic* if $d_i > 1$ and aperiodic if $d_i = 1$.

A state i is *recurrent* if with probability 1, we return back:

$$f_{i,i} := P(\exists t > 0 : X_t = i \mid X_0 = i) = 1.$$

A state i is *transient* if it is not recurrent.

1. A spider and a fly move along a straight line in unit increments. The spider always moves towards the fly by one unit. The fly moves towards the spider by one unit with probability 0.3, moves away from the spider by one unit with probability 0.3, and stays in place with probability 0.4. The initial distance between the spider and the fly is an integer. When the spider and the fly land in the same position, the spider captures the fly. (They may "exchange positions" though.)

(a) Construct a Markov chain that describes the relative location of the spider and fly.

(b) Identify the transient and recurrent states.

2. Consider a Markov chain with states 1, 2, …, 9. and the following transition probabilities: $p_{1,2} = p_{1,7} = 1/2$, $p_{i,i+1} = 1$ for $i \neq 1, 6, 9$ and $p_{6,1} = p_{9,1} = 1$. Determine which states are recurrent/transient/periodic.

3. Let C be a communicating class, i.e., an equivalence class of $\leftrightarrow$.

(a) Either all states in C are recurrent or all are transient.

(b) All states in C have the same period.

4. Consider a finite irreducible Markov chain (X_t) with at least two states and its two properties:

(a) (X_t) is aperiodic.

(b) There is n such that for every i, j we have $r_{i,j}(n) > 0$.

Show that (b) implies (a). Think for a while about the other implication (but a proper proof is somewhat lengthy).

Stationary distribution of Markov chains

Recall: every *finite irreducible aperiodic* Markov chain converges to a stationary distribution. The stationary distribution π satisfies the conditions

$$\sum_{j \in S} \pi_j = 1 \qquad \text{and} \qquad \sum_i \pi_i p_{i,j} = \pi_j \quad (\text{for all } j \in S)$$

called the *balance equations.*

5. Consider the example with broken/working machine from the class. What is the stationary distribution? Does the Markov chain converge to it?

You may solve this either by solving a system of equations or by sampling the process on computer, or by multiplying matrices. Try all of the possibilities.

6. Consider a Markov chain that is finite, irreducible and aperiodic. Assume that for $n \geq 50$, the n-step transition probabilities are very close to the steady-state probabilities i.e., the stationary distribution). So, in particular, for every i we have $P(X_{100} = j \mid X_0 = i) \doteq \pi_j$.

(a) Find approximately $P(X_{100} = j, X_{101} = k \mid X_0 = i)$.

(b) Find approximately $P(X_{100} = j, X_{101} = k, X_{200} = \ell \mid X_0 = i)$.

(c) Find approximately $P(X_{100} = i \mid X_{101} = j)$.

Verify what you have found on the working/broken machine example.

7. A *birth-death process (proces zrodu a zániku)* is a Markov chain with states $0, 1, \dots, m$ and transitions

- $p_{i,i+1} = b$ for $i < m$
- $p_{i,i-1} = d$ for $i > 0$
- $p_{0,0} = 1 - b$
- $p_{m,m} = 1 - d$
- $p_{i,i} = 1 - b - d$ for $0 < i < m$

Write the balance equations and find the stationary distribution.

8. (Ehrenfest model of diffusion.) We have a total of n balls, some of them black, some white. At each time step, we either do nothing, which happens with probability ε, where $0 < \varepsilon < 1$, or we select a ball at random, so that each ball has probability $(1 - \varepsilon)/n > 0$ of being selected. In the latter case, we change the color of the selected ball (if white it becomes black, and vice versa), and the process is repeated indefinitely. What is the steady-state distribution of the number of white balls?

9. (Bernoulli-Laplace model of diffusion.) Each of two urns contains m balls. Out of the total of the $2m$ balls, m are white and m are black. A ball is simultaneously selected from each urn and moved to the other urn, and the process is indefinitely repeated. What is the steady-state distribution of the number of white balls in each urn?

10. * (Local balance equations.) We are given a finite irreducible aperiodic Markov chain. Suppose that we have found a solution π to the following system of local balance and normalization equations:

$$\sum_{j \in S} \pi_j = 1 \qquad \pi_i p_{i,j} = \pi_j p_{j,i} \quad \text{(for all } i, j \in S\text{)}$$

(a) Show that the π_js are the steady-state probabilities.

(b) What is the interpretation of the equations in terms of expected long-term frequencies of transitions between i and j?

(c) Construct an example where the local balance equations are not satisfied by the steady-state probabilities.

11. Define a particular way of shuffling cards as a Markov chain. (Suggestion: we take a couple of cards from the top of the stack at put them somewhere in the middle without changing their order.) Is the Markov chain aperiodic? Irreducible?