Home assignment 2

Probabilistic techniques 2

Submission deadline: Apr 11 before class for full credit, one week later with a penalty
Discussion of solutions: Apr 18 after class (as scheduled)

Only problems marked with (*) are to be submitted. The rest are practice problems. To get credit you need 50 % of the points.

1(*). Let $G = (V, E)$ be the graph whose vertices are all 7^n vectors of length n over $\mathbb{Z}_7$, in which two vertices are adjacent iff they differ in precisely one coordínate. Let $U \subset V$ be a set of 7^{n-1} vertices of G, and let W be the set of all vertices of G whose distance from U exceeds $(c+2)\sqrt{n}$, where $c > 0$ is a constant. Prove that $|W| \leq 7^n e^{-c^2/2}$.

2(*). Let $G = (V, E)$ be a graph with chromatic number $\chi(G) = 1000$. Let $U \subset V$ be a random subset of V chosen uniformly among all $2^{|V|}$ subsets of V. Let $H = G[U]$ be the induced subgraph of G on U. Prove that

$$Pr[\chi(H) < 400] < 1/100.$$

3(*). Prove that there is an absolute constant c such that for every $n > 1$ there is an interval I_n of at most $c\sqrt{n}/\log n$ consecutive integers such that the probability that the chromatic number of $G(n, 0.5)$ lies in I_n is at least 0.99.

4(*). Let $T(G)$ be the number of triangles in the graph G. For $G \sim G(n,p)$ find what concentration bound can be obtained by using each of: Azuma, Talagrand, Kim-Vu.

5(*). For a permutation π, let $F(\pi)$ be the number of fixed points, i.e., number of x such that $\pi(x) = x$. Choose π uniformly at random from all permutations of $\{1, \ldots, n\}$. Find a concentration estimate for $F(\pi)$. Use Azuma and/or Talagrand. Preferably both, and compare.

6. In the proof of Talagrand we used the following estimate:

$$e^{(1-\lambda)^2/4} r^{-\lambda} \leq 2 - r$$

for all $r \in [0, 1]$ and appropriate $\lambda = \lambda(r)$. Verify that this is true.

7. Two questions to think about regarding the proof of Talagrand:

- Why do we have $e^{-t^2/4}$ in the bound and not the usual $e^{-t^2/3}$?

- In the proof we defined the set $U(A,x)$ as the set of all $s \in \{0,1\}^n$ such that

$$\exists y \in A : x_i \neq y_i \implies s_i = 1$$

It would be more natural to have an equivalence here, as it would more directly correspond to the definition of $\varrho(A,x)$. Can you see, where it would fail?

8. Let X, A be sets. Let d_a for $a \in A$ be a metric. Then the function $d(x,y) \mapsto \sup_{a\in A} d_a(x,y)$ is also a metric. (We used similar function to prove Talagrand inequality.)

9.
- Azuma implies Chernoff (for $\mathrm{Bi}(n,p)$) when $p = 1/2$ – but not in general.
- As does Talagrand but with worse constants.

10. For each martingale $X_0, \dots, X_m$ we have $\mathbb{E}[X_j \mid X_i] = X_i$ for every $j \geq i$ (*). The opposite direction is false: there is a sequence of random variables $X_0, \dots, X_m$, which meets (*), but does not constitute a martingale.

11. In the proof of Azuma we have used the following properties of conditional expectation (where we are conditioning on another random variable).

- $\mathbb{E}[\mathbb{E}[X \mid Y]] = \mathbb{E}[X]$
- $\mathbb{E}[\mathbb{E}[X \mid Y, Z] \mid Z] = \mathbb{E}[X \mid Z]$
- $\mathbb{E}[\mathbb{E}[f(X)g(X,Y) \mid X]] = \mathbb{E}[f(X)\,\mathbb{E}[g(X,Y) \mid X)]]$

For reference

- **Azuma's inequality** Let $X_0, \dots, X_m$ be a martingale s.t. $|X_{k+1} - X_k| \leq c_k$. Then for every $t > 0$

$$\Pr[X_m \leq X_0 - t] < e^{-\frac{t^2}{2\sum_{k=0}^{n-1} c_k^2}}$$
$$\Pr[X_m \geq X_0 + t] < e^{-\frac{t^2}{2\sum_{k=0}^{n-1} c_k^2}}$$

- **Azuma's inequality – corollary** Let X be c-Lipschitz on $\Omega = \prod_{i=1}^n \Omega_i$. Then for every $t > 0$

$$\Pr[X \leq \mathbb{E}[X] - t\sqrt{n}] < e^{-t^2/(2c^2)}$$
$$\Pr[X \geq \mathbb{E}[X] + t\sqrt{n}] < e^{-t^2/(2c^2)}$$

- **Talagrand's inequality I** Let A be a subset of the product probability space $\Omega = \prod_i \Omega_i$. Then $\Pr[X \in A]\Pr[X \notin A_t] \leq e^{-t^2/4}$, where A_t is the set of such x that $\rho(x,A) \leq t$, and ρ is supremum over all unit vectors α of $\inf_{y\in A} \sum_{i:x_i\neq y_i} \alpha_i$.

- **Talagrand's inequality II** Let X be c-Lipschitz and f-certifiable. Then for any b, t (where $t \geq 0$)

$$\Pr[X \leq b - t\sqrt{f(b)}]\Pr[X \geq b] \leq \exp(-\frac{t^2}{2c^2})$$

- **Talagrand's inequality III** Let X be c-Lipschitz and r-certifiable. (This means f-certifiable for $f(s) = rs$.) Then for any $0 \leq t \leq \mathbf{Med}[X]$

$$\Pr[|X - \mathbf{Med}[X]| > t] \leq 4\exp(-\frac{t^2}{8c^2 r\,\mathbf{Med}[X]})\,.$$

- **Talagrand's inequality IV** Let X be c-Lipschitz and r-certifiable. (This means f-certifiable for $f(s) = rs$.) Then for any $0 \leq t \leq \mathbb{E}[X]$

$$\Pr[|X - \mathbb{E}[X]| > t + 60c\sqrt{r\,\mathbb{E}[X]}] \leq 4\exp(-\frac{t^2}{8c^2 r\,\mathbb{E}[X]})\,.$$

- **Kim-Vu inequality** Let H be a hypergraph, $w : E(H) \to [0, \infty)$, $t_i \sim Bern(p_i)$ for $i \in V(H)$. Put

$$Y = \sum_{e \in E(H)} w_e \prod_{i \in e} t_i \qquad \text{and}$$

$$Y_A = \sum_{e \in E(H): e \supseteq A} w_e \prod_{i \in e - A} t_i \qquad \text{for } A \subseteq V(H)$$

Let $E_i = \max\{\mathbb{E}[Y_A| : |A| = i\}$. (Note, that $E_0 = \mathbb{E}[Y]$.) Further, let $E' = \max\{E_i : 1 \leq i \leq k\}$ and $E = \max\{E_i : 0 \leq i \leq k\}$. Then for any $\lambda > 1$ we have

$$\Pr[|Y - \mathbb{E}[Y]| > a_k\sqrt{EE'}\lambda^k] < d_k e^{-\lambda} n^{k-1},$$

where $a_k = 8^k\sqrt{k!}$ and $d_k = 2e^2$.