Home assignment 1

Probabilistic techniques 2

Submission deadline: Mar 14 before class
Discussion of solutions: Mar 21 after class (as scheduled)

Only problems marked with (*) are to be submitted. The rest are practice problems. To get credit you need 50 % of the points.

Most of the problems relate directly to the lecture and enhance your understanding of it. On the other hand, the last three problems are there to test your mastery of the presented techniques on a fresh problem. We strongly suggest you attempt some of them. (However, no hard rule will be imposed.)

1. Using the Harris-Kleitman lemma (for down-sets as stated in the lecture) prove that any two up-sets $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ satisfy

$$|\mathcal{A} \cap \mathcal{B}| \geq \frac{|\mathcal{A}||\mathcal{B}|}{2^n}.$$

State and prove a corresponding statement for an up-set and a down-set.

2(*). For a family $\mathcal{A} \subseteq 2^{[n]}$ define

$$\mathcal{A} - \mathcal{A} := \{A \setminus A' : A, A' \in \mathcal{A}\}.$$

Prove that for any $\mathcal{A}$ we have $|\mathcal{A} - \mathcal{A}| \geq |\mathcal{A}|$.

Hint: In the Four functions theorem set $\alpha = \beta = \gamma = \delta \equiv 1$. Now, given $\mathcal{A}$, find an appropriate $\mathcal{B}$.

3(*). Derive the Harris-Kleitman lemma directly from the Four functions theorem.

4. Prove the $n = 1$ case of the Four functions theorem.

5. Let L be a distributive lattice (recall: this means $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z \in L$). Prove that

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{1}$$

for all $x, y, z \in L$. Deduce that a lattice is distributive if and only if (1) holds for any three elements.

6. Prove that any finite distributive lattice L is isomorphic to a subset of a boolean cube partially ordered by inclusion.

Hint: Call an element $x \in L$ *join-irreducible* if whenever $x = y \vee z$ then either $x = y$ or $x = z$. Use the set of all join-irreducible elements in L to construct the desired isomorphism.

7(*). Let $(P, \leq)$ be a finite poset and and $x, y, z \in P$ be such that the pairs (x, y) and (x, z) are incomparable. Prove from the first principles (that is, without invoking the XYZ-theorem) that there exists a linear extension of P in which $x \leq y \wedge x \leq z$ holds.

8. Prove that that the poset L introduced in the proof of the XYZ-theorem is a lattice.

9(*). Let $\mu : L \to \mathbb{R}^+$ be the (normalized) measure introduced in the proof of the XYZ-theorem, and let $\mathcal{A} = \{(x_1, \ldots, x_n) : x_i \neq x_j \text{ for all } 1 \leq i < j \leq n\}$. Prove that μ conditioned on $\mathcal{A}$ is the uniform measure on the linear extensions of P.

10. In some probability space, let A, B, C, D be events of positive probability, satisfying $\mathbb{P}(A)\mathbb{P}(B) \leq \mathbb{P}(C)$ and $\mathbb{P}(D) \geq 1 - \epsilon$, for some $0 < \epsilon$.

$$\mathbb{P}(A \mid D)\mathbb{P}(B \mid D) \leq \mathbb{P}(C \mid D)(1 + o_{\epsilon \to 0}(1))$$

11. Show that if $\mathcal{A}$ is an up-set and $\mathcal{B}$ is down-set then $\mathcal{A} \,\square\, \mathcal{B} = A \cap B$. Show that if $\mathcal{C}$ and $\mathcal{D}$ are up-sets then so is $\mathcal{C} \,\square\, \mathcal{D}$.

12. Let $\mathcal{A}$ and $\mathcal{B}$ be up-sets on $[n]$, and let $\mathcal{C} = \mathcal{A} \,\square\, \mathcal{B}$. Let $\mathcal{A}^- = \{A \subseteq [n-1] \colon A \in \mathcal{A}\}$, $\mathcal{A}^+ = \{A \subseteq [n-1] \colon A \cup \{n\} \in \mathcal{A}\}$, and define $\mathcal{B}^-, \mathcal{B}^+, \mathcal{C}^-, \mathcal{C}^+$ analogously. Show that $\mathcal{C}^- = \mathcal{A}^- \,\square\, \mathcal{B}^-$ and $\mathcal{C}^+ = (\mathcal{A}^- \,\square\, \mathcal{B}^+) \cup (\mathcal{B}^- \,\square\, \mathcal{A}^+)$.

13(*). Investigate the equality cases in the Harris-Kleitman lemma and the BK theorem.

14(*). Let G be a graph and let P denote the probability that a random subgraph of G obtained by picking each edge of G with probability $1/2$, independently, is connected (and spanning). Let Q denote the probability that in a random two-colouring of G, where each edge is chosen, randomly and independently, to be either red or blue, the red graph and the blue graph are both connected (and spanning). Is $Q \leq P^2$?

15(*). A family of subsets $\mathcal{G}$ is called *intersecting* if $G_1 \cap G_2 \neq \emptyset$ or all $G_1, G_2 \in \mathcal{G}$. Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k$ be k intersecting families of subsets of $\{1, 2, \ldots, n\}$. Prove that

$$\left| \bigcup_{i=1}^{k} \mathcal{F}_i \right| \leq 2^n - 2^{n-k}.$$

16(*). Show that the probability that in the random graph $G(2k, 1/2)$ the maximum degree is at most $k - 1$ is at least $1/4^k$.