Combinatorics and Graph Theory III - 2020/21

Series 9

1. Let p be a prime. Suppose that average degree of a multigraph G is greater than $2p-2$. Show, that G has a nonempty submultigraph, in which every vertex has degree divisible by p. If $\Delta(G) \leq 2p-1$, then G contains a p-regular subgraph.

2. We call a graph *almost d-regular* if all vertices have degree d or $d+1$. Show that if G is almost d-regular, but no subgraph of G is almost d-regular, then G has a matching covering all vertices of degree $d+1$.

3. Show that if G is almost d-regular, then G has a spanning subgraph that is almost $(d-1)$-regular but not $(d-1)$-regular.

4. Show that if G is an almost d-regular graph for $d \geq 4$ and G is not 4-regular, then G has 3-regular subgraph.

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5. Find a connected graph G with an orientation with all in-degrees 2, coefficient of $x_1^2 x_2^2 \dots x_n^2$ in its polynomial is 0, but still G is 3-choosable.

6. Let $v_1, \dots, v_n$ be an ordering of vertices of G such that $v_1 v_2 \in E(G)$ and for $i \geq 3$ vertex v_i has exactly two neighbors among $\{v_1, \dots, v_{i-1}\}$ Let

$$p_G = \prod_{v_i v_j \in E(G), i<j} (x_j - x_i)$$

be the graph polynomial of G in variables $x_1, \dots, x_n$. For any function $f : [n] \to N$ let $c(f, g)$ be the coefficient of $x_1^{2-f(1)} x_2^{2-f(2)} \dots x_n^{2-f(n)}$ in p_G. (If $f(x) > 2$ for some $x \in [n]$ then $c(f, G) = 0$.) Let $e_i : [n] \to N$ be a function such that $e_i(i) = 1$ and $e_i(x) = 0$ for $x \in [n] \setminus \{i\}$. Write $\widetilde{c}(f, G) = c(f + e_2, G) - c(f + e_1, G)$.

Show by induction over $n \geq 2$, that if f satisfies $\sum_{x \in [n]} f(x) = 2$, then $\widetilde{c}(f, G) \equiv 1 \pmod{3}$.

7. Let G be 2-degenerated, $v_0 \in V(G)$, L a list assignments such that $|L(v)| \geq 3$ for $v \in V(G) \setminus \{v_0\}$ and $|L(v_0)| = 1$. Show that G is L-colorable.

8. Let G be a graph with a Hamilton cycle K such that $G - E(K)$ is a collection of vertex-disjoint triangles. Show that G is 3-choosable.