Combinatorics and graph theory 3 – 2020/21
Series 7

1. Show that
$$\lim_{n\to\infty} t_{r-1}(n)/\binom{n}{2} = \frac{r-2}{r-1}.$$

2. Assume Erdős-Stone theorem. Prove from it Erdős-Stone-Simonovits theorem:
$$\lim_{n\to\infty} ex(n,H)/\binom{n}{2} = 1 - \frac{1}{\chi(H)-1}.$$

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3. * "common neighborhood" Let (A,B) be an ε-regular pair (in some graph) with $d(A,B)=d$, let $s>0$ be an integer. For a tuple $\vec{a}=(a_1,\ldots,a_s)\in A^s$ we let $N(\vec{a})=\cap_{i=1}^{s} N(a_i)$ be the common neighborhood of vertices in $\vec{a}$. Let the set $Y\subseteq B$ satisfy $(d-\varepsilon)^{s-1}|Y|\geq \varepsilon|B|$. Then
$$|\{\vec{a}\in A^s : |Y\cap N(\vec{a})| < (d-\varepsilon)^s|Y|\}| < s\varepsilon|A|^s\,.$$

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4. Let $|A|=|B|=|C|=n$, let (A,B), (B,C), (C,A) be three ε-regular pairs, for some $\varepsilon\in(0,1/2]$. Let $t=t(A,B,C)$ the number of triangles with one vertex in A, another in B and the third in C. Then
$$|t-d(A,B)d(B,C)d(C,A)n^3|\leq 13\varepsilon n^3.$$

5. Suppose G is a graph and (A,B) is an ε-regular pair in G for some $0<\varepsilon\leq 1$. Let $|A|=|B|=n$ and $p=d(A,B)$. Show that the number of 4-cycles $v_1v_2v_3v_4$ v G s.t. $v_1,v_3\in A$ and $v_2,v_4\in B$ is at least $p^4n^4-17\varepsilon n^4-2n^3$.

6. Show that for every $p>0$ there are $c,\varepsilon>0$ such that the following is true: Let G be a grpah and (A,B) an ε-regular pair in G. Assume $|A|=|B|=n$ and $d(A,B)\geq p$. Let $A'\subseteq A$ and $B'\subseteq B$ be sets such that $|A'|=|B'|\geq(1-\varepsilon)n$, every vertex of A' has at least $(p-2\varepsilon)n$ neighbors in B' and every vertex of B' has at least $(p-2\varepsilon)n$ neighbors in A'. Then the bipartite subgraph of G with parts A', B' has at least cn mutually edge disjoint pefect matchings.

7. Prove the following: for every $\alpha>0$ there are $c,n_0>0$ such that each graph G with $n\geq n_0$ vertices and at least αn^2 edges has $\lceil cn\rceil$-regular bipartite graph as a subgraph.

8. * Let $0\leq p\leq 1$ be real. Show that if $A_1,\ldots,A_4$ are disjoint subsets of vertices of some graph G of the same size n, pairs (A_i,A_j) are ε-regular for $1\leq i<j\leq 4$, $d(A_i,A_j)\geq p$ for $(i,j)\in\{(1,2),(2,3),(3,4),(4,1)\}$ and $d(A_i,A_j)\leq p$ for $(i,j)\in\{(1,3),(2,4)\}$, then G has at least $p^4(1-p)^2n^4-100\varepsilon n^4$ induced 4-cycles.