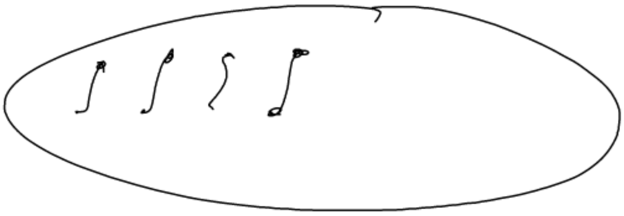
Packing & covering

- pack <u>vertex</u> disjoint copies of K_2

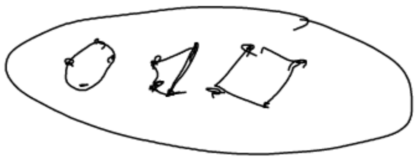


matching

- Hall theorem
- Tutte theorem

cond. for general graphs

- <u>packing of vertex disjoint cycles</u>



<u>Erdős-Pósa thm</u>

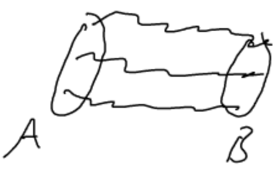
$\exists f: \mathbb{N} \to \mathbb{N}$ s.t. $\forall k \, \forall G$ either G has k vertex disj. cycles

or $\exists X \subseteq V(G)$: $G - X$ has no cycle & $|X| \le f(k)$

G # of cycles in G $\le |X|$

(not true with $f(k) = k$, unfortunately ... <u>exercise</u>)

- pack edge disj. paths $\Longleftrightarrow$ k-edge connection (Menger's thm)



$|A| = |B| = k$

- pack vertex disj. paths $\Leftrightarrow$ k-vertex conn.
- packing dir. paths

  max-flow = min cut

Packing & covering by spann. trees

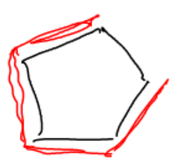
(1) How many edge-disjoint spanning trees does a graph G have? <u>MAX</u> DST

(2) How many sp. trees do we need to cover E(G)? <u>MIN</u> STC

Motivation:
- nice :)
- k disj. sp. trees $\Rightarrow$ $\forall u,v$ k edge disj. u-v paths & eff. algo

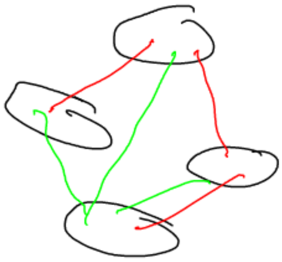
k edge disj. sp.t's. $\Rightarrow$ k-edge conn.

$\not\Leftarrow$



k-DST $\Longrightarrow$ $\forall$ part'n $V(G) = V_1 \cup \ldots \cup V_r$

$\forall$ sp. tree has $\geq r-1$ edges in between



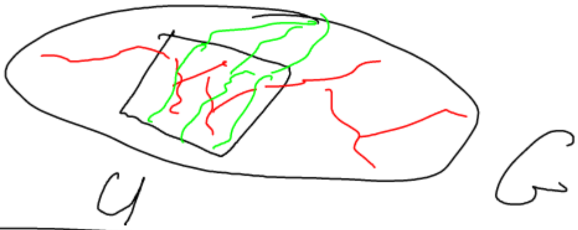
$\Longrightarrow$ $\underset{\substack{\| \\ |E(G)|}}{\|G\|} \geq k(r-1)$,

G has $\geq k(r-1)$ edges in-between.

Thm 1 (Nash-Williams, Tutte 1961)

Multigraph G (conn.) has k-DST $\Longleftrightarrow$ $\forall$ partition (of $V(G)$) into r parts $\geq k(r-1)$ edges in between

k-STC $\Longrightarrow$ $\forall U \subseteq V(G)$ $\|G[U]\| \leq k(|U|-1)$



Thm 2 (Nash-Williams 1964)

Multigraph G has (conn.) STC by $\leq k$ trees

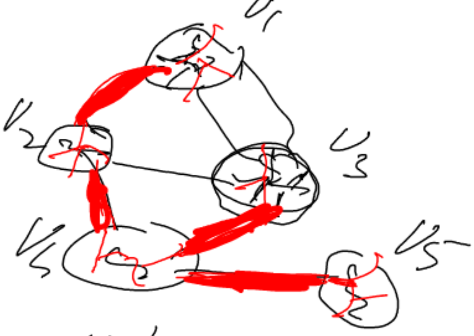
$\Longleftrightarrow$ $\forall U \subseteq V(G)$ $\|G[U]\| \leq k(|U|-1)$

(Def arboricity of G = min k s.t. $E(G)$ can be covered by k sp-trees)

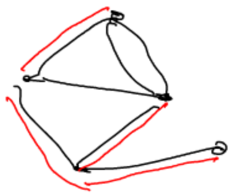
$\boxed{\text{Thm 3}}$ [Bowler, Carmesin, 2015] $\forall G$ conn. multigraph $\forall k$ $\exists$ partition P of $V(G)$ s.t.

① $\forall U \in P$ $G[U]$ has k edge disj. sp. trees

② G/P has k-STC.



$P = \{V_1, \dots, V_5\}$, G/P



$P = \{V_1 \dots V_5\}$

$\boxed{\text{Thm } 3 \Rightarrow \text{Thm } 2}$

G multigraph, conn.

---> P from Thm 3

Assume $\forall U : \|G[U]\| \leq \underline{k(|U|-1)}$ edges

$U = V_i$ ⟶ $G[U]$ has k edg. disj. sp. tr.

$\} \Rightarrow \|G[U]\| = k(|U|-1)$

$G[U]$ is a $\boxed{\text{union of } k \text{ e.d.s.t.}}$

$T_1^i \dots T_k^i$

② $\Rightarrow$ sp. trees in G/P $T_1 \dots T_k$

$T_1' = T_1 \cup T_1^1 \cup \dots \cup T_1^r$ is a sp. tree in G

T_2'

$\vdots$

T_k'

cover of $E(G)$

(edges of G/P ... by Thm 3 ②

edges in V_i

$\boxed{\text{Thm } 3 \Rightarrow \text{Thm } 1}$

G, P ---> G/P — by Thm 3, ② --- $T_1, \dots, T_k$ k-STC

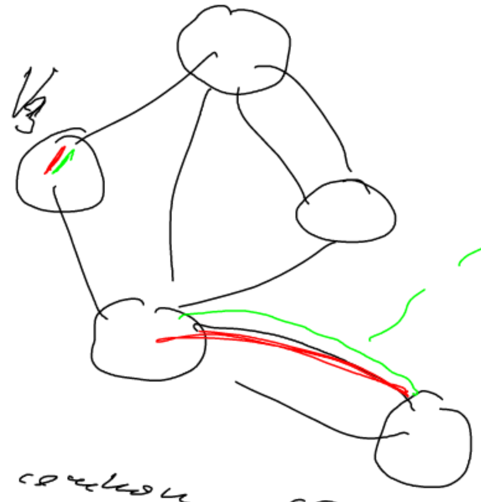
by ass. of Thm 1 : $\|G/P\| \geq k(r-1)$

$\} \Rightarrow$ $\|G/P\| = k(r-1)$ & $T_1 \dots T_k$ are k-DST in G/P

$G[V_i]$ has $T_1^i, \dots, T_k^i$ k-DST $\Longleftarrow$ (*)

$T_1' = T_1 \cup T_1^1 \cup \dots \cup T_1^r$ is a sp. tree in G

$T_2' \dots T_k'$ ——— disjoint sp. trees



$T_1' \cap T_2'$ has no edge in G/P

T_1^3 & T_2^3 have no edge in common ⟶ (*)

T — sp. tree of G e — chord ($e \notin E(T)$)

C_e — fund. cycle of e w.r.t. T (the cycle of $T+e$)

$f \in C_e$ ⟶ $T' = T+e-f$ another sp. tree

$\mathcal{T} = (T_1, \dots, T_k)$ ⟶ $E(\mathcal{T}) = \bigcup\{E(T) : T \in \mathcal{T}\}$

seq. $e_0, e_1, \dots, e_n \in E(G)$ <u>exchange chain</u> for $\mathcal{T}$ <u>started by e_0</u>

1) $\forall i < n$ $\exists j = j(i)$ s.t. $e_i \in T_j$, e_{i+1} is a chord of T_j, its fund. cycle wrt T_j contains e_i

2) $\boxed{e_n \notin E(T_1) \cup \dots \cup E(T_k)}$

$i = 0$

$e_0 \in T_j$

e_1

$j(0) = 2$

$T_2^1 := T_2^0 + e_1 - e_0$

Lemma If e_0 starts an exchange chain for $\mathcal{T}$ & <u>lies in two of its trees</u>
then $\exists$ family $\mathcal{T}'$ of sp. trees s.t. $E(\mathcal{T}) \subsetneq E(\mathcal{T}')$

- choose exch. chain $e_0, e_1, \ldots, e_n$ of min. length (given e_0)
- no e_i lies on fund. cycle wrt. any tree in $\mathcal{T}$ of any e_ℓ, $\ell > i+1$ (owise: $e_0 \ldots e_i, e_\ell, e_{\ell+1} \ldots e_n$ a shorter ex. chain)
- $\mathcal{T}^0 = \mathcal{T}$; $\mathcal{T}^{i+1}$ $(i = 0, \ldots, n-1)$

$$\left\{ \begin{array}{l} \mathcal{T}^i = (T_1^i, \ldots, T_k^i) \\ \mathcal{T}^{i+1} = (T_1^{i+1}, \ldots, T_k^{i+1}) \end{array} \right. \qquad \begin{array}{l} T_j^{i+1} := T_j^i + e_{i+1} - e_i \text{ for } j = j(i) \\ T_j^{i+1} = T_j^i \text{ owise} \end{array}$$

- e_0 is in two trees $\Rightarrow$ we delete it from one, repl. by e_1 (now e_1 is in two trees)

  ..... $E(\mathcal{T}^n) = E(\mathcal{T}^0) \cup \{e_n\}$

(Proof of Thm 3) $\mathcal{T} = (T_1, \ldots, T_k)$ family of k sp. trees of G, $E(\mathcal{T})$ is maximal

$D = \{e \in E(G) : e \text{ starts an exch. chain for } \mathcal{T}\}$

$\forall e \notin E(\mathcal{T}) : e \in D$ .... ex. ch. with $n = 0$

P ... part. of $V(G)$ into components of $(V(G), D)$

① $U \in P$ $\quad S_j$ = subgraph of $T_j[U] \cap D$ ----

$S_1 \dots S_k$ ... forests

— edge disj.: (if $e \in S_1 \cap S_2$ .... then $e \in D$

$e_0 = e$ starts an ex. ch.

by Lemma ... we find larger J')

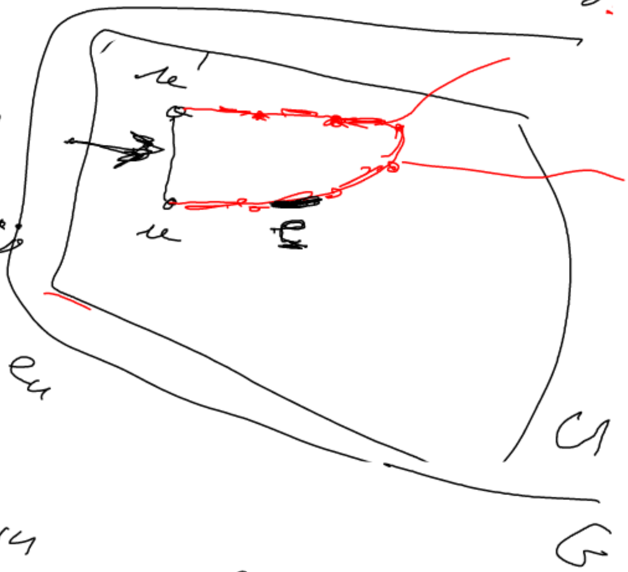
?, S_i connected?

→ $E(D)$ form a conn. subgraph of $G[U]$

→ suffices to show $uu' \in D$, $u, u' \in U$ → $\exists$ u-u' path in S_j

1) $uu' \in T_j \Rightarrow uu' \in S_j$ ✓

2) $uu' \notin T_j$ path uT_ju' has all ? its edges in D ⇒ all in S_j ✓



$e_0 = uu' \in D$ ⇒ ∃ ex. ex. chain $e_0, e_1, \dots, e_n$

we want: $e \in D$

$e, e_0, e_1, \dots, e_n$ is an exch. chain

② recall: $U \in P$ $\quad T_j[U]$ is a tree (S_j)

contracting U to a vertex, T_j stays a tree

$T_1 \dots T_k / P$ — k-tuple of (sp.) trees in G/P $(T_1', \dots, T_k')$

want: $T_1' \dots T_k'$ cover G/P

[$e \in E \setminus E(\mathcal{T}) \Rightarrow e \in D \Rightarrow \exists U \in P$: e has both ends in $U \Rightarrow$ in G/P e is contracted

Thm 3 [Bowler, Carmesin, 2015] $\forall G$ conn. multigraph $\forall k$ $\exists$ partition P of $V(G)$ st.

① $\forall U \in P$ $G[U]$ has k edge disj. sp. trees

② G/P has k-STC.