Proof of Reg. Lemma

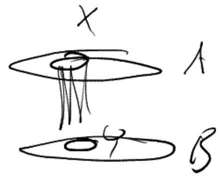
(acc. to Alon–Spencer, Prob. Method)

<u>Thm</u> (Reg. Lemma, Szemerédi 1978)

$\forall \varepsilon > 0 \; \forall m \; \exists M = M(\varepsilon, m) \; \forall G : |G| \geq M \Rightarrow G$ has ε-reg. part. into $\{V_0, \dots, V_k\}$, $m \leq k \leq M$.

<u>Proof idea</u> start with any partition

(when we see irregularity, we refine)



if $d(X,Y)$ is not close to $d(A,B) \Rightarrow$ refine

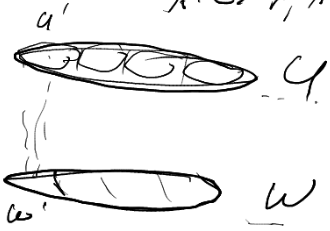
$A \rightsquigarrow X, A \setminus X, \quad B \rightsquigarrow Y, B \setminus Y$

def. $U, W \subseteq V$

$$q(U,W) = \frac{|U| \cdot |W|}{n^2} \underbrace{d(U,W)}_{\in [0,1]}^2 = \frac{\|U,W\|^2}{|U| \cdot |W| \cdot n^2}$$

$\mathcal{U}$ ... part. of U, $\mathcal{W}$ ... part. of W

$$q(\mathcal{U}, \mathcal{W}) = \sum_{\substack{U' \in \mathcal{U} \\ W' \in \mathcal{W}}} q(U', W')$$



P ... part. with exc. set V_0 $\rightsquigarrow \bar{P} = \{\{x\}, x \in V_0\} \cup \{V_1, V_2, \dots, V_k\}$

$\{V_0, V_1, \dots, V_k\}$

$$q(P) := q(\bar{P}) = \sum_{\substack{U < W \\ U, W \in \bar{P}}} \underbrace{q(U,W)}_{\leq \frac{|U| \cdot |W|}{n^2}} \quad \dots \text{ index of } P$$

$= q(V_1, V_2) + q(V_1, V_3) + \dots$

$$\leq \frac{1}{n^2} \sum |U| \cdot |W| \leq \frac{\binom{n}{2}}{n^2} \leq \frac{1}{2}$$

Lemma

1) disj., nonempty $U, W \subseteq V$; $\mathcal{U}, \mathcal{W}$ ... part. of U, W

Then $q(\mathcal{U}, \mathcal{W}) \geq q(U, W)$

2) P', P ... partitions of V, P' is refinement of P $\Rightarrow q(P') \geq q(P)$

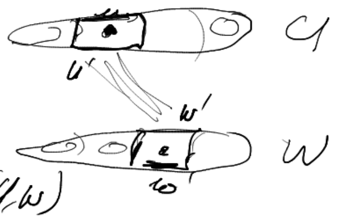
3) $\varepsilon > 0$, U, W as in 1), (U, W) is not ε-regular

$\Rightarrow \exists\, \mathcal{U} = (U_1, U_2)$ of U, $\mathcal{W} = (W_1, W_2)$ of W s.t. $\boxed{q(\mathcal{U}, \mathcal{W}) \geq q(U, W) + \varepsilon^4 \frac{|U| \cdot |W|}{n^2}}$

partitions

① def. r.v. Z: $u \in_u U$, $w \in_u W$ ... $U' \in \mathcal{U}$, $W' \in \mathcal{W}$ s.t. $U' \ni u$ & $W' \ni w$

unif. random element $\quad Z = d(U', W')$



$$\mathbb{E} Z = \sum_{\substack{U' \in \mathcal{U} \\ W' \in \mathcal{W}}} d(U', W') \frac{|U'|}{|U|} \cdot \frac{|W'|}{|W|} = \sum_{U', W'} \frac{\|U', W'\|}{|U| \cdot |W|} = \frac{\|U, W\|}{|U| \cdot |W|} = d(U, W)$$

Jensen ineq.: $\mathbb{E} Z^2 \geq (\mathbb{E} Z)^2$

$$\mathbb{E} Z^2 = \sum_{U', W'} d^2(U', W') \cdot \frac{|U'| \cdot |W'|}{|U| \cdot |W|} = \underbrace{\sum \frac{d^2(U', W') \cdot |U'| \cdot |W'|}{n^2}}_{q(\mathcal{U}, \mathcal{W})} \cdot \frac{n^2}{|U| \cdot |W|}$$

$\sum$ of $q(U', W')$

$\mathbin{\rotatebox{90}{$\geq$}}$

$$(\mathbb{E} Z)^2 = \underbrace{\frac{d^2(U, W)\, |U| \cdot |W|}{n^2}}_{q(U, W)} \cdot \frac{n^2}{|U| \cdot |W|} = q(U, W) \frac{n^2}{|U| \cdot |W|}$$

② easy conseq. of ①

sum this over all pairs U, W $\longrightarrow$ $q(U, W) \leq q(\mathcal{U}, \mathcal{W})$

$\mathcal{U} = \{P \in P' ;\ P \subseteq U\}$

$\mathcal{W} = \{P : P \subseteq W\}$

$$q(P) \leq q(P')$$

③ as (U, W) is not ε-reg., there are $U_1 \subseteq U$ & $W_1 \subseteq W$

$P(|Z - \mathbb{E}Z| > \varepsilon) \geq \frac{|U_1|}{|U|} \cdot \frac{|W_1|}{|W|} \geq \varepsilon^2$

$|U_1| \geq \varepsilon |U|$, $|W_1| \geq \varepsilon |W|$

$|d(U_1, W_1) - d(U, W)| > \varepsilon$

$U_2 := U \setminus U_1$, $W_2 := W \setminus W_1$

$\mathcal{U} = (U_1, U_2)$, $\mathcal{W} = (W_1, W_2)$

Z as in part 1

$\operatorname{Var} Z = \mathbb{E}(Z^2) - (\mathbb{E} Z)^2 = \frac{n^2}{|U| \cdot |W|} \cdot (q(\mathcal{U}, \mathcal{W}) - q(U, W))$

$\mathbb{E} Z = d(U, W)$

$= \operatorname{Var}(Z) = \mathbb{E}(Z - \mathbb{E}Z)^2 \geq \varepsilon^2 \cdot \varepsilon^2 = \varepsilon^4$

$q(\mathcal{U}, \mathcal{W}) - q(U, W) \geq \varepsilon^4 \cdot \frac{|U| \cdot |W|}{n^2}$

Prop. $0<\varepsilon\le\frac{1}{4}$, $P=\{V_0,V_1,\dots,V_k\}$ equipartition $\quad |V_1|=|V_2|=\dots=|V_k|=c$, $|V_0|\le\varepsilon n$, $n=|G|$

IF P is NOT ε-regular **THEN** $\exists P'$ refin. of P $\quad P'=\{V_0',V_1',\dots,V_\ell'\}$

s.t. $k\le\ell\le k\cdot 4^k$, $|V_0'|\le|V_0|+\frac{n}{2^k}$ $\quad \overline{P'}$ is a refine. of $\overline{P}$

$|V_1'|=\dots=|V_\ell'|$ & $\boxed{q(P')\ge q(P)+\frac{1}{2}\varepsilon^5}$

Pf → $\forall\, 1\le i<j\le k$ def. $\mathcal{V}_{ij}$ of V_i, $\mathcal{V}_{ji}$ of V_j — (V_i,V_j) is ε-regular: $\mathcal{V}_{ij}=\{V_i\}$, $\mathcal{V}_{ji}=\{V_j\}$

(V_i,V_j) not ε-reg. $\Rightarrow$ use Lemma 3, $|\mathcal{V}_{ij}|=|\mathcal{V}_{ji}|=2$

→ $\forall i=1,\dots,k$: $\mathcal{V}_i :=$ common refinement of $\mathcal{V}_{i1},\mathcal{V}_{i2},\dots,\mathcal{V}_{ik}$ $\quad \mathcal{V}_{ii}=\{V_i\}$

$|\mathcal{V}_i|\le 2^{k-1}$ $\quad Q:=\bigcup\mathcal{V}_i\cup\{V_0\}$

$$q(Q)\ge q(P)+\underbrace{\varepsilon k^2}_{\#(i,j):\ (V_i,V_j)\text{ not }\varepsilon\text{-regular}}\ \underbrace{\varepsilon^4\frac{c^2}{n^2}}_{\text{increase from L.3}}\ \ge$$

$q(P)=\sum_{i,j} q(V_i,V_j)$ (+ exc. set)

total incr. $\quad$ incr. by $\ge\varepsilon^4\frac{c^2}{n^2}$ if (V_i,V_j) not ε-reg.

$\sum_{i,j} q(\mathcal{V}_{ij},\mathcal{V}_{ji})$

increase — L.1

$\sum_{i,j} q(\mathcal{V}_i,\mathcal{V}_j)$

$kc=|G|-|V_0|\le n$ $\quad \frac{k^2c^2}{n^2}\le 1$

$\ge n-\varepsilon n=(1-\varepsilon)n\ge\frac{3}{4}n$

$q(Q)\ge q(P)+\varepsilon^5\cdot\left(\frac{3}{4}\right)^2\ge \boxed{q(P)+\frac{\varepsilon^5}{2}}$

$|Q|\le k2^{k-1}$ + exc. set

Missing: make the non-exceptional parts equal

$b:=\lfloor\frac{c}{4^k}\rfloor$ split each part of Q arbitrary into sets of size b

with the remainder put to V_0' $\quad\leadsto P'$

big sets $\le\frac{c\cdot k}{b}=k4^k$, all of size b

$|V_0'|\le|V_0|+(b-1)\cdot k\cdot 2^{k-1}\le|V_0|+\frac{c}{4^k}\cdot k\cdot 2^k=|V_0|+\frac{n}{2^k}$

$\overline{P'}$ is a ref. of $\overline{Q}$, so $q(P')\ge q(Q)\ge q(P)+\frac{\varepsilon^5}{2}$

Proof of the Reg. Lemma

WMA $\varepsilon \le \frac{1}{4}$, m satisfies $\quad 2^{m-2} > \frac{1}{\varepsilon^6}$

$s = \lceil \frac{1}{\varepsilon^5} \rceil \qquad \boxed{\frac{1}{2^k} \le \frac{\varepsilon}{2s}} \qquad \forall k \ge m \qquad s \le \varepsilon \cdot 2^{k-1}$

$\left(s = \lceil \frac{1}{\varepsilon^5} \rceil < \frac{2}{\varepsilon^6} \cdot \varepsilon < 2\varepsilon \cdot 2^{m-2} = \qquad \frac{1}{2^m} < \frac{2\varepsilon}{4s} \right)$

$k_0 = m$, $k_{i+1} = k_i 4^{k_i}$; $\quad M := k_s \quad >> 4^{s^{s^{\cdot^{\cdot^{m}}}}} \Big\} \lceil \frac{1}{\varepsilon^5} \rceil$ - times

G; $|G| = n \ge M$, $\quad P = P_0$ ... any part. into $k = k_0 = m$ parts of equal size $\lfloor \frac{n}{m} \rfloor$, $|V_0| \le m \ll \frac{1}{2}\varepsilon n$

If P_i is not ε-regular, we use Prop. to refine it & get P_{i+1}

$\le k_i$ parts $\qquad q(P_{i+1}) \ge q(P_i) + \frac{\varepsilon^5}{2} \qquad \le k_{i+1}$ parts

& the size of exc. set increases by $\le \frac{n}{2^{k_i}} \le \frac{\varepsilon}{2s} \cdot n$

$\frac{1}{2} \ge q(P_i) \ge q(P_0) + i \cdot \frac{\varepsilon^5}{2} \ge i \frac{\varepsilon^5}{2} \implies i \le s$

the final exc. set $\le |V_0| + s \cdot \frac{\varepsilon}{2s} n \le \frac{1}{2}\varepsilon n + \frac{1}{2}\varepsilon n \le \varepsilon n$

after $\le s$ steps we stop $\implies P_i$ is ε-regular for some $i \le s$.

the # of parts is $\le k_i \le k_s \overset{df.}{=} M$